Course Material
Real Analysis
I M.Sc., Mathematics
UNIT I
Continuity and compactness

A mapping f of a set E in R^k is said to be bounded if there is a real number M such the $|f(x)| \le M$ for all $x \in E$.

Theorem 1.1.8:

Suppose f is a continuous mapping of a compact metric spaces X into a metric space Y. Then f(X) is compact.

Proof:

Let $\{V_\alpha\}$ be an open cover of f(X).

Since f is continuous,

Theorem 1.1.5, shows that each of the sets $f^{-1}(V_\alpha)$ is open.

Since X is compact,

There are finitely many indices, say $\alpha_1, ..., \alpha_n$ such that $X \subset f^{-1}(V_{\alpha 1}) \cup ... \cup f^{-1}(V_{\alpha n})$ ----------(1)

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, (1) implies that

$f(X) \subset V_{\alpha 1} \cup .... \cup V_{\alpha n}$ ----------(2)

Hence proved.

Theorem 1.1.9:

If f is a continuous mapping of a compact metric space X into R^k, then f(X) is closed and bounded. Thus f is bounded.

Theorem 1.1.10:

Suppose f is a continuous real function on a compact metric space D,and

$M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$ ---------(1)

Then there exist points $p, q \in X$ such that f(p)=M and f(q)=m.

Here M – the least upper bound of the set of all numbers f(p), where p ranges over X.

m-the greatest lower bound of this set of numbes.

Conclusion:

There exist points p and q in X such that $f(q) \le f(x) \le f(p)$ for all $x \in X$;

(i.e.,) f attains its maximum (at p) and its minimum (at q).

Proof:

By theorem 1.1.9,

f(X) is closed and bounded set of real numbers;

here f(X) contains

M=sup f(X) and m = inf f(X).

[by previous theorems, Let E be a nonempty set of real numbers which is bounded above .Let y= sup E.

Then $y \in \overline{E}$.

Hence $y \in E$ if E is closed.]

Theorem 1.1.11:

Suppose f is continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x, \quad x \in X$$

is continuous mapping on Y onto X.

Proof:

Applying theorem 1.1.5 to f^{-1} in place of f, it suffices to prove that f(V) is an open set in Y for every open set V in X.

Fix such a set V.

The complement V^c of V is closed in X,

Hence compact. [since closed subsets of compact sets are compact]

Hence $f(V^c)$ is a compact subset of Y. [since theorem 1.1.8]

So is closed in Y. [since compact subsets of metric spaces are closed]

Since f is 1-1 and onto,

$f(V)$ is the complement of $f(V^c)$.

Hence f(V) is open.

Uniformly Continuous

Let f be a mapping of a metric space X into a metric space Y. we say that f is uniformly continuous on X if for every $\in > 0$ there exists $\delta > 0$ such that

$$d_Y\left(f(p), f(q)\right) < \in$$

For all p and q in X for which $d_X\left(p, q\right) < \delta$

Differences between continuity and uniform continuity

Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.

A given function is uniform continuous at a certain point is meaningless.

If f is continuous on X, then it is possible to find, for each $\in > 0$ and for each point p of X, a number $\delta > 0$ having the property specified in the definition of continuous functions. This δ depends on $\in$ and on p.

If f is however, uniformly continuous on X, then it is possible, for each $\in > 0$ to find one number $\delta > 0$ which will do for all points p of X.

Every uniformly continuous function is continuous.

Theorem 1.1.12:

Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

Proof:

Let $\in > 0$ be given.

Since f is continuous,

We can associate to each point $p \in X$ a positive number $\phi(p)$ such that

$$q \in X,\ d_X(p,q) < \phi(p)$$

$$\Rightarrow d_Y(f(p), f(q)) < \frac{\in}{2} \quad \text{--------(1)}$$

Let J(p) be the set of all $q \in X$ for which

$$d_X(p,q) < \frac{1}{2}\phi(p) \quad \text{---------(2)}$$

Since $p \in J(p)$, the collection of all sets J(p) is an open cover of X;

Since X is compact, there is a finite set of points $p_1, \ldots, p_n$ in X, such that

$$X \subset J(p_1) \cup \ldots \cup J(p_n) \quad \text{--------(3)}$$

We put $\delta = \frac{1}{2}\min\left[\phi(p_1), \ldots, \phi(p_n)\right]$ ______(4)

Then $\delta > 0$

Now let q and p be points of X,

Such that

$$d_X(p,q) < \delta$$

By (3), there is an integer m, $1 \le m \le n$,

Such that $p \in J(p_m)$;

Hence $d_X(p, p_m) < \frac{1}{2}\phi(p_m)$ -------(5)

And we have,

$$d_X(q, p_m) \le d_X(p,q) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \le \phi(p_m)$$

Finally, (1) shows that

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)) < \in$$

This completes the proof.

Theorem 1.1.13:

Let E be a non-compact set in R^1. Then

(a) There exists a continuous function on E which is not bounded.

(b) There exists a continuous and bounded function on E which has no maximum.

If, in addition, E is bounded, then

(c) There exists a continuous function of E which is not uniformly continuous.

Proof:

Suppose first that E is bounded,

So that there exists a limit point x_0 of E which is not a point of E.

Consider $f(x)=\dfrac{1}{x-x_0}, \quad x\in E$ --------(1)

This is continuous on E (theorem 1.1.6), but evidently unbounded.

To see that (1) is not uniformly continous,

Let $\in>0$ and $\delta>0$ be arbitrary, and

Choose a point $x\in E$ such that $|x-x_0|<\delta$.

Taking t close enough to x_0,

We can then make the difference $|f(t)-f(x)|$ greater than $\in$,

although $|t-x|<\delta$,

since this is true for every $\delta>0$,

f is not uniformly continuous on E.

The function g given by $g(x)=\dfrac{1}{1+(x-x_0)^2}, \; x\in E$ --------(2)

Is continuous on E, and is bounded, since $0<g(x)<1$.

It is clear that $\sup\limits_{x\in E} \; g(x)=1,$

Whereas g(x)<1 for all $x\in E$.

Thus g has no maximum on E.

Having proved the theorem for bounded sets E,

Let us now suppose that E is bounded,

Then f(x)=x establishes (a),

Whereas $h(x)=\dfrac{x^2}{1+x^2}, x\in E$ -------(3)

Establishes (b), since

$$\sup_{x\in E} \; h(x)=1,$$

And h(x)<1 for all $x\in E$.

Assertion (c) would be false if boundedness were omitted from the hypotheses.

For, let E be the set of all integers.

Then every function defined on E is uniformly continuous on E.

To see this,

We need merely take $\delta>1$in definition uniformly continuous.

We prove this section by showing that compactness is also essential in theorem 1.1.11.

Example:

Let X be the half-open interval $[0,2\pi)$ on the real line, and let f be the mapping of X onto the circle Y consisting of all points whose distance from the origin is 1, given by

$f(t) = (\cos t, \sin t), \quad 0 \le t < 2\pi$ -------(1)

The continuity of the trigonometric functions cosine and sine and their periodicity properties, f is a continuous 1-1 mapping of X onto Y.

However, the inverse mapping fails to be continuous at the point (1, 0)=f(0).

X is not compact in this example.

Continuity and connectedness

Theorem 1.1.14:

If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then f(E) is connected.

Proof:

Let us assume, on the contrary,

That $f(E) = A \cup B$,

Where A and B are nonempty separated subsets of Y.

Put $G = E \cap f^{-1}(A)$,

$H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \overline{A}$ (the closure of A),

We have $G \subset f^{-1}(\overline{A})$;

The latter set is closed, since f is continuous;

$\overline{G} \subset f^{-1}(\overline{A})$

It follows that $f(\overline{G}) \subset \overline{A}$.

Since f(H)=B and $\overline{A} \cap B$ is empty,

$\overline{G} \cap H$ is empty.

The same argument show that $\overline{G} \cap H$ is empty.

Thus G and H are separated.

This is impossible if E is connected.

Theorem 1.1.15:

Let f be a continuous real function on the interval [a,b]. If f(a) < f(b) and if c is a number such that f(a)<c<f(b), then there exists a point $x \in (a, b)$ such that f(x)=c.

Definition and Existence of the integral

Let [a, b] be a given interval,

By a partition P of [a, b] we mean a finite set of points $x_0, x_1, ..., x_n$, where

$a = x_0 \le x_1 \le ... \le x_{n-1} \le x_n = b$

we write

$$\Delta x_i = x_i - x_{i-1}\ ,\ \ i=1,2,...,n$$

Suppose f is a bounded real function defined on [a, b].

Corresponding to each partition P of [a, b]

We put

$$M_i = \sup\ f(x),\ \ x_{i-1} \le x \le x_i$$

$$m_i = \inf\ \ f(x),\ \ x_{i-1} \le x \le x_i$$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

And $\overline{\int_a^b} f dx = \inf\ \ U(P,f)$ ----------(1)

$\underline{\int_a^b} f dx = \sup\ \ L(P,f)$ ----------(2)

Where the inf and the sup are taken over all partitions P of [a, b].

The left members of (1) & (2) are called the upper and lower Riemann integrals of f over [a, b] respy.

If the upper and lower integrals are equal, we say that f is Riemann-integrable on [a, b],

We write $f \in \Re$ ($\Re$ denotes the set of Riemann-integrable functions)

We denotes the common value of (1) and (2) by

$\int_a^b f\ \ dx$ ---------(3) (or)

$\int_a^b f(x)\ dx$ ------(4)

This is the Riemann integral of f over [a, b].

Since f is bounded, there exist two numbers, m and M such that

$$m \le f(x) \le M, \; a \le x \le b.$$

Hence for every P,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

So that the numbers L(P, f) and U(P, f) form a bounded set.

This shows that the upper and lower integral are defined for every bounded function f.

Definition

Let α be a monotonically increasing function on [a, b]. [since α(a) and α(b) are finite, it follows that α is bounded on [a, b]]

Corresponding to each partition of P of [a, b],

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$\Delta\alpha_i \ge 0$$

For any real function f which is bounded on [a, b].

Corresponding to each partition P of [a, b]

We put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \,,$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

And we define $\int_a^{\bar{b}} f d\alpha = \inf \; U(P,f,\alpha)$ ----------(5)

$$\int_{\underline{a}}^{b} f d\alpha = \sup \; L(P,f,\alpha) \quad \text{----------(6)}$$

The left members of (5) & (6) are equal,

$$\int_a^b f\ d\alpha \quad \text{---------(7) (or)} \quad \int_a^b f(x)\ d\alpha(x) \quad \text{------(8)}$$

we say that f is Riemann-Stieltjes integral of f with respect to α,over [a, b],

If (5) & (6) are equal we say that f is integrable w.r.t. α, in the Riemann sense, $f \in \Re(\alpha)$

Partition:

The partition P* is called as a refinement of P if $P^* \supset P$. Given two partitions, P_1 and P_2, we say that P* is their common refinement if $P^* = P_1 \cup P_2$

Theorem 1.2.1.:

If P* is a refinement of P,

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \text{ ---------(1)}$$

And $\quad U(P^*, f, \alpha) \leq U(P, f, \alpha)$ ----------(2)

Proof:

To prove (1), suppose first that P* contains one point more than P.

Let this extra point be x*,

And suppose $x_{i-1} < x^* < x_i$,

Where x_{i-1} and x_i are two consecutive points of P,

Put w_1= inf f(x) , $\quad x_{i-1} \leq x \leq x^*$

W_2=inf f(x), $\quad x^* \leq x \leq x_i$

Clearly $w_1 \geq m_i$ and $w_2 \geq m_i$,

Hence

$$L(P^*, f, \alpha) - L(P, f, \alpha) = w_1\left[\alpha(x^*) - \alpha(x_{i-1})\right] + w_2\left[\alpha(x_i) - \alpha(x^*)\right] - m_i\left[\alpha(x_i) - \alpha(x_{i-1})\right]$$

$$= (w_1 - m_i)\left[\alpha(x^*) - \alpha(x_{i-1})\right] + (w_2 - m_i)\left[\alpha(x_i) - \alpha(x^*)\right] \geq 0$$

If P*contains k points more than P, we repeat this reasoning k times, and arrive at (1). The proof of (2) is analogous.

Theorem 1.2.2:

$$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$$

Proof:

Let P* be the common refinement of two partitions P_1 and P_2.

By previous theorem,

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$$

Hence $\quad L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$ --------(1)

If P_2 is fixed and the sup is taken over all P_1,

(1) Gives $\underline{\int} f \; d\alpha \leq U(p_2, f, \alpha)$ -----------(2)

The theorem follows by taking the inf over all P_2 in (2).

Theorem 1.2.3:

$f \in \Re(\alpha)$ on [a, b] if and only if for every ε>0 there exists a partition P such that

$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ ----------(1)

Proof:

For every P we have

$$L(P, f, \alpha) \leq \underline{\int} f \; d\alpha \leq \overline{\int} f \; d\alpha \leq U(P,f,\alpha)$$

Thus (1) implies $\overline{\int} f \; d\alpha - \underline{\int} f \; d\alpha < \varepsilon$

Hence if (1) can be satisfied for every $\varepsilon > 0$,

We have $\overline{\int} f \; d\alpha = \underline{\int} f \; d\alpha$, that is $f \in \Re(\alpha)$.

Conversely,

Suppose $f \in \Re(\alpha)$,

And let $\varepsilon > 0$ be given.

Then there exist partitions P_1 and P_2 such that

$$U(P_2, f, \alpha) - \int f \, d\alpha < \varepsilon/2 \qquad \text{------(2)}$$

$$\int f \, d\alpha - L(P_1, f, \alpha) < \varepsilon/2 \qquad \text{------(3)}$$

We choose P to be the common refinement of P_1 and P_2.

Then theorem 1.2.1, together with (2) & (3) shows that

$$U(P, f, \alpha) \le U(P_2, f, \alpha) < \int f \, d\alpha + \varepsilon/2 < L(P_1, f, \alpha) + \varepsilon \le L(P, f, \alpha) < \varepsilon$$

So that (1) holds for this partition P.

Theorem 1.2.4:

If f is monotonic on [a, b], and if α is continuous on [a, b], then $f \in \Re(\alpha)$.

Proof:

let $\varepsilon > 0$ be given.

For any positive integer n, choose a partition such that $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$, i=1,2,...,n.

This is possible since α is continuous . [Theorem 1.1.15]

We suppose that f is monotonically increasing.

Thus $M_i = f(x_i)$, $m_i = f(x_i - 1)$, i=1,2,...,n

So that

$$U(P, f, \alpha) - L(P, f, \alpha) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1}) \right]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)] < \varepsilon$$

If n is taken enough.

By theorem 1.2.3, $f \in \Re(\alpha)$.

Theorem 1.2.5:

If f is continuous on [a, b] then $f \in \Re(\alpha)$ on [a, b].

Proof:

let $\varepsilon > 0$ be given.

Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \varepsilon.$$

Since f is uniformly continuous on [a, b] [theorem 1.1.12]

There exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta \qquad \text{------(1)}$$

If $x \in [a, b]$, $t \in [a,b]$, and $|x - t| < \delta$,

If P is any partition on [a, b] such that $\Delta x_i < \delta$ for all i, then (1) implies that

$$M_i - m_i \leq \eta, \text{ (i-1, ...,n)}$$

And therefore,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$\leq \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon$$

By theorem 1.2.3, $f \in \Re(\alpha)$.

Theorem 1.2.6:

Suppose f is bounded on [a, b], f has only finitely many points of discontinuity on [a, b], and α is continuous at every point at which f is discontinuous. Then $f \in \Re(\alpha)$.

Proof:

let $\varepsilon > 0$ be given,

put $M = \sup |f(x)|$,

Let E be the set of points at which f is discontinuous.

Since E is finite and α is continuous at every point of E,

We can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ such that the sum of the corresponding differences $\alpha(v_j) - \alpha(u_j)$ is less that ε.

We can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from [a, b].

The remaining set K is compact.

Hence f is uniformly continuous on K, and there exists δ>0 such that

$$|f(s) - f(t)| < \varepsilon \text{ if } s \in K,\ t \in K,\ |s - t| < \delta.$$

Now from a partition $P = \{x_0, x_1, ..., x_n\}$ of [a, b],

Each u_j occurs in P, Each v_j occurs in P.

No point of any segment (u_j, v_j) occurs in P.

If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Note that $M_i - m_i \le 2M$ for every i, and that $M_i - m_i \le \varepsilon$ unless x_{i-1} is one of the u_j.

Hence as in the proof of theorem 1.2.5,

$$U(P, f, \alpha) - L(P, f, \alpha) \le [\alpha(b) - \alpha(a)]\varepsilon + 2M\varepsilon$$

Since ε is arbitrary,

Theorem 1.2.3, shows that $f \in \Re(\alpha)$.

Theorem 1.2.7:

Suppose $f \in \Re(\alpha)$ on [a, b], $m \le f \le M$, φ is continuous on [m, M], and h(x)= φf(x) on [a, b]. Then $f \in \Re(\alpha)$ on [a, b].

Proof:

Choose ε>0,

Since φ is uniformly continuous on [m, M],

There exists δ>0 such that δ<ε and $|\varphi(s)-\varphi(t)| < \varepsilon$ if $|s-t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in \Re(\alpha)$,

There is a partition $P = \{x_0, x_1, ..., x_n\}$ of [a, b],

Such that $U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$ ----------------(1)

Let M_i, m_i have the same meaning in the definition.

Let M_i*, m_i* be the analogous numbers for h.

Divide the numbers 1,2,…,n into two classes:

$i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$

For $i \in A$, out choice of δ shows that

$$M_i * - m_i * \le \varepsilon,$$

For $i \in B$, $M_i * - m_i * \le 2k$,

Where $k = \sup|\phi(t)|$, $m \le t \le M$,

By (1), we have

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \delta^2 \text{ ------------(2)}$$

So that $\sum_{i \in B} \Delta\alpha_i < \delta$

It follows that

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=A} (M_i * - m_i *)\Delta\alpha_i + \sum_{i=B} (M_i * - m_i *)\Delta\alpha_i$$

$$\leq \varepsilon\left[\alpha(b)-\alpha(a)\right]+2k\delta < \varepsilon\left[\alpha(b)-\alpha(a)+2k\right]$$

Since ε was arbitrary.

Theorem 1.2.3 implies that $h \in \Re(\alpha)$

Properties of the Integral

(a) If $f_1 \in \Re(\alpha)$ and $f_2 \in \Re(\alpha)$ on [a, b], then $f_1+f_2 \in \Re(\alpha)$,

$$\int_a^b (f_1+f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

If $cf \in \Re(\alpha)$

Then $\int_a^b cfd\alpha = c\int_a^b fd\alpha$

(b) If $f_1(x) \leq f_2(x)$ on [a, b], then

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

(c) If $f \in \Re(\alpha)$ on [a, b] and a<c<b, then $f \in \Re(\alpha)$ on [a, c] and on [c, b],

and $\int_a^c fd\alpha + \int_c^b fd\alpha = \int_a^b fd\alpha$

(d) If $f \in \Re(\alpha)$ on [a, b] and if $\left|f(x)\right| \leq M$ on [a, b] then

$$\left|\int_a^b fd\alpha\right| \leq M\left[\alpha(b)-\alpha(a)\right]$$

(e) If $f \in \Re(\alpha_1)$ and $f \in \Re(\alpha_2)$, then $f \in \Re(\alpha_1+\alpha_2)$, and

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b fd\alpha_1 + \int_a^b f_2 d\alpha_2$$

If $f \in \Re(\alpha)$ and c is a positive constant, then $\int_a^b f\, d(c\alpha) = c\int_a^b fd\alpha$

Theorem 1.2.8:

If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on [a, b],

Then (a) $fg \in \Re(\alpha)$

(b) $|f| \in \Re(\alpha)$ and $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$

Proof:

If we take $\phi(t) = t^2$,

Theorem 1.2.7, shows that $f^2 \in \Re(\alpha)$ if $f \in \Re(\alpha)$.

The identity

$$4fg = (f+g)^2 - (f-g)^2$$

Completes the proof of (a).

If we take $\phi(t) = |t|$,

Theorem 1.2.7. shows that $|f| \in \Re(\alpha)$.

Choose $c = \pm 1$, so that

$$\left|\int_a^b f\, d\alpha\right| = c\int_a^b f d\alpha = \int_a^b c f d\alpha \le \int_a^b |f|\, d\alpha$$

Since $cf \le |f|$

Unit Step function:

The unit step function I is defined by $I(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$.

Theorem 1.2.9:

If a<s<b, f is bounded on[a, b], f is continuous at s, and $\alpha(x) = I(x - s)$ then,

$$\int_a^b f\,d\alpha = f(s)$$

Proof:

Consider partitions $P = \{x_0, x_1, x_2, x_3\}$,

Where x_0=a and x_1=s<x_2<x_3=b.

Then U(P,f,α)=M_2, L(P,f,α)=m_2,

Since f is continuous at s, we see that M_2 and m_2 converge to f(s) as $x_2 \to s$.

Theorem 1.2.10:

Suppose $C_n \geq 0$ for 1,2,3,…., $\sum C_n$ converges, {s_n}is a sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} C_n I(x - s_n) \quad \text{----------(1)}$$

Let f be continuous on [a, b], then

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} C_n f(s_n) \quad \text{-------------(2)}$$

Proof:

The comparison test shows that the series (1) converges for every x.

Its sum α(x) is evidently monotonic, and α(a)=0, $\alpha(b) = \sum Cn$

Let ε>0 be given, and choose N so that $\sum_{N+1}^{\infty} C_n < \varepsilon$

Put

$$\alpha_1(x) = \sum_{n=1}^{N} C_n I(x - s_n),\ \alpha_2(x) = \sum_{N+1}^{\infty} C_n I(x - s_n)$$

By properties of integral and theorem 1.2.9,

$$\int_a^b f\,d\alpha_1 = \sum_{n=1}^{N} C_n f(s_n) \quad \text{-------(3)}$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$,

$$\left|\int_a^b f\, d\alpha_2\right| \le M\varepsilon \quad \text{----------(4)}$$

Where $M = \sup |f(x)|$.

Since $\alpha = \alpha_1 + \alpha_2$,

It follows from (3) & (4),

$$\left|\int_a^b f\, d\alpha - \sum_{n=1}^{N} C_n f(s_n)\right| \le M\varepsilon$$

If we let $N \to \infty$, we obtain (2).

Theorem 1.2.11:

Assume α increases monotonically and $\alpha' \in \Re$ on [a, b]. Let f be a bounded real function on [a, b].

Then $f \in \Re(\alpha)$ iff $f\alpha' \in \Re$

In that case $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)dx$

Theorem 1.2.12:

Suppose ϕ is a strictly increasing continuous function that maps an interval [A, B] on [a, b]. suppose α is monotonically increasing on [a, b] and $f \in \Re(\alpha)$ on [a, b].

Define β and g on [A, B] by

$$\beta(y) = \alpha(\phi(y)), \qquad g(y) = f(\phi(y)) \quad \text{--------- (1)}$$

Then $g \in \Re(\beta)$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha \quad \text{-------------(2)}$$

Proof:

To each partition $P=\{x_0,...,x_n\}$ of [a, b] corresponds a partition $Q=\{y_0,...,y_n\}$ of [A, B],

So that $x_i=\phi(y_i)$.

All partitions of [A, B] are obtained in this way.

Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$,

We see that

$$U(Q,g,\beta)=U(P,f,\alpha),\ L(Q,g,\beta)=L(P,f,\alpha) \quad \text{--------(3)}$$

Since $f\in\Re(\alpha)$, P can be choosen so that both $U(P,f,\alpha)$ and $L(P,f,\alpha)$ are close to $\int f\ d\alpha$.

Hence (3), combined with theorem 1.2.3, shows that $g\in\Re(\beta)$ and that (2) holds.

This completes the proof.

Integration and Differentiation:

Theorem 1.2.13:

Let $f\in\Re(\alpha)$ on [a, b].

For $a\le x\le b$,

Put $F(x)=\int_a^x f(t)\ dt$.

Then F is continuous on [a, b];

Furthermore, if f is continuous at a point x_0 of [a, b], then F is differentiable at x_0,

And $F'(x_0)=f(x_0)$

Proof:

Since $f\in\Re$, f is bounded.

Suppose $|f(t)|\le M$ for $a\le t\le b$.

If $a \le x < y \le b$ then,

$$|F(y) - F(x)| = \left|\int_x^y f(t)dt\right| \le M(y - x),$$

By the properties of the integral,

Given ε>0,

We see that $|F(y) - F(x)| < \varepsilon$,

Provided that $|y - x| < \varepsilon / M$.

This proves continuity of F.

Now suppose f is continuous at x_0.

Given ε>0, choose δ>0 such that

$$|f(t) - f(x_0)| < \varepsilon$$

If $|t - x_0| < \delta$, and $a \le t \le b$.

Hence if

$$x_0 - \delta < s \le x_0 \le t < x_0 + \delta \text{ and } a \le s < t \le b,$$

By the properties of the integral (d),

$$\left|\frac{F(t) - F(s)}{t - s} - f(x_0)\right| = \left|\frac{1}{t - s}\int_s^t [f(u) - f(x_0)]dx\right| < \varepsilon$$

It follows that $F'(x_0) = f(x_0)$.

Theorem 1.2.14: The Fundamental theorem of calculus

If $f \in \Re$ on [a, b] and if there is a differentiable function F on [a, b] such that $F' = f$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof:

Let ε>0 be given.

Choose a partition $P=\{x_0,...,x_n\}$ of [a, b]

So that U(P, f)= L(P, f)<ε.

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i)-F(x_{i-1})=f(t_i)\Delta x_i$$

For i=1,2,…,n,

Thus

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = F(b)-F(a)$$

From the known theorem, it follows that

$$\left| F(b)-F(a)-\int_a^b f(x)\ dx \right| < \varepsilon$$

Since this holds for every ε>0, the proof is complete.

UNIT II

Definition:

Suppose $\{f_n\}$, n=1,2,3,…, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad (x \in E) \text{. ---------(1)}$$

Here we say that $\{f_n\}$ converges on E and that f is the limit, or the limit function, of $\{f_n\}$.

Some times we say that "$\{f_n\}$ converges to f pointwise onE" if (1) holds.

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad (x \in E) \text{ --------(2)}$$

The function f is called the sum of the series $\sum f_n$.

f is continuous at a limit point x if

$$\lim_{t\to\infty} f(t) = f(x)$$

Limit of a sequence of continuous functions is continuous.

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(x) \text{ ---------(3)}$$

Example 1:

For m=1, 2, 3, …, n, n=1, 2, 3, …, let $S_{m,n} = \dfrac{m}{m+n}$

Then for every fixed n,

$$\lim_{m\to\infty} S_{m,n} = 1,$$

So that $\lim_{n\to\infty}\lim_{m\to\infty} S_{m,n} = 1,$ -----------------(1)

On the other hand, for every fixed m,

$$\lim_{n\to\infty} S_{m,n} = 0,$$

So that

$$\lim_{n\to\infty}\lim_{m\to\infty} S_{m,n} = 0, \quad \text{-------(2)}$$

Example 2:

$$\text{Let } f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad (x \text{ real}, n=1,2,3,...) \text{ -------(1)}$$

And $f(x) = \lim_{n\to\infty} f_n(x) = 0$

Then $f^{'}(x) = 0$, and

$$f_n^{'}(x) = \sqrt{n}\cos nx,$$

So that {fn'} does not converge to f'.

For instance, $f_n^{'}(0) = \sqrt{n} \to \infty$

As $n \to \infty$, whereas f'(0)=0.

Example 3:

$$f_n(x) = n^2 x(1-x^2)^n, \quad (0 \le x \le 1, \ n=1,2,3,...) \text{ -------(1)}$$

For $0 < x \le 1$, we have

$$\lim_{n\to\infty} f_n(x) = 0,$$

By known theorem,

Since $f_n(0) = 0$,

$$\lim_{n\to\infty} f_n(x) = 0, \ (0 \le x \le 1) \quad \text{--------(2)}$$

$$\int_0^1 x(1-x^2)^n dx = \frac{1}{2n+2}$$

In spite of (2),

$$\int_0^1 f_n(x)dx = \frac{n^3}{2n+2} \to +\infty \text{ as } n \to \infty$$

If in (1), we replace n^2 by n, (2) still holds,

But we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2},$$

Whereas

$$\int_0^1 \left[\lim_{n\to\infty} f_n(x) \right] dx = 0$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

Uniform Convergence

A sequence of functions $\{f_n\}$, n=1,2,3,…, converges uniformly on E to a function f if for every ε>0 there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon \quad \text{for all } x \in E. \text{ -------(1)}$$

Every uniformly convergent sequence is point wise convergent.

The difference between the two concept:

- If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every ε>0, and for every $x \in E$, there is an integer N, depending on ε and on x, such that (1) holds if $n \geq N$;
- if $\{f_n\}$ converges uniformly on E, it is possible, for each ε>0, to find one integer N which will do for all $x \in E$.

The series $\sum f_n(x)$ converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{t=1}^{n} f_t(x) = s_n(x)$$

Converges uniformly on E.

Theorem 2.1: Cauchy Criterion for Uniform Convergence

The sequence of functions { f_n} defined on E, converges uniformly on E if and only if for every ε>0 there exists an integer N such that $m \geq N,\ x \in E$ implies

$$|f_n(x) - f_m(x)| \leq \varepsilon \text{ --------(1)}$$

Proof:

Suppose { f_n} converges uniformly on E,

Let f be the limit function.

Then there is an integer N such that $n \geq N,\ x \in E$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2},$$

So that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

If $n \geq N,\ m \geq N,\ x \in E$.

Conversely,

Suppose the Cauchy condition holds.

The sequence $\{f_n(x)\}$ converges, for every x, to a limit which we may call f(x).

Thus the sequence $\{f_n\}$ converges on E, to f.

Now we have to prove that convergence is uniform.

Let ε>0 be given,

Choose N such that (1) holds,

Fix n, and let $m \to \infty$ in (1).

Since $f_m(x) \to f(x)$ as $m \to \infty$,

That implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

For every $n \geq N$ and every $x \in E$, Which completes the proof.

Theorem 2.2:

Suppose $\lim_{n\to\infty} f_n(x) = f(x),\ (x \in E)$.

Put $M_n = \sup_{x\in E} \left\| f_n(x) - f(x) \right\|$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Theorem 2.3:

Suppose $\{f_n\}$ is a sequence of functions defined on E, and

Suppose $\left|f_n(x)\right| \le M_n,\quad (x \in E,\ n=1,2,3,...)$.

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.[converse is not true]

Proof: If $\sum M_n$ converges, then,

For arbitrary ε>0,

$$\left|\sum_{t=n}^{m} f_t(x)\right| \le \sum_{t=n}^{m} M_t \le \varepsilon,\quad (x \in E),$$

Provided m and n are large enough. Uniform convergence now proved from theorem 2.1.

UNIFORM CONVERGENCE AND CONTINUITY

Theorem 2.4:

Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that

$$\lim_{t\to x} f_n(t) = A_n,\ (n=1,2,3,...)\quad \text{---------(1)}$$

Then $\{A_n\}$ converges, and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n,\ (n=1,2,3,...)\ \text{------(2)}$$

In other words,

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t),\ \ (n=1,2,3,...) \text{------}(3)$$

Proof:

Let ε>0 be given.

By the uniform convergence of $\{f_n\}$, there exists N such that $n \ge N, m \ge N, t \in E$ implies

$$|f_n(t) - f_m(t)| \le \varepsilon \quad \text{---------}(4)$$

Let $t \to x$ in (4), we obtain

$$|A_n - A_m| \le \varepsilon$$

For $n \ge N, m \ge N,$ so that $\{A_n\}$ is a Cauchy sequence and therefore converges , to A(say).

$$|f(t) - A| \le |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \quad \text{--------}(5)$$

We first choose n such that

$$|f(t) - f_n(t)| \le \frac{\varepsilon}{3} \quad \text{---------}(6)$$

For all $t \in E$, and such that

$$|A_n - A| \le \frac{\varepsilon}{3} \quad \text{---------}(7)$$

Then for this n, we choose a neighborhood V of x such that

$$|f_n(t) - A_n| \le \frac{\varepsilon}{3} \quad \text{--------}(8)$$

If $t \in V \cap E,\ t \ne x$.

Substituting the inequalities (6) to (8) into (5),

We see that $|f(t) - A| \le \varepsilon$

Provided $t \in V \cap E,\ t \ne x$.

This is equivalent

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$$

Theorem 2.5:

If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Note:

The converse is not true; that is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform.

Theorem 2.6:

Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, n=1,2,3,…

Then $f_n \to f$ uniformly on K.

Proof:

Put $g_n = f_n - f$.

Then g_n is continuous,

$g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$.

We have to prove that $g_n \to 0$ uniformly on K.

Let ε>0 be given.

Let K_n be the set of all $x \in K$ with $g_n(x) \geq \varepsilon$.

Since g_n is continuous,

K_n is closed.

Hence compact. [since known theorem]

Since $g_n \geq g_{n+1}$,

We have $K_n \supset K_{n+1}$.

Fix $x \in K$.

Since $g_n(x) \to 0$,

We see that $x \notin K_n$ if n is sufficiently large.

Thus if $x \notin \cap K_n$.

In other words, $\cap K_n$ is empty.

Hence K_n is empty for some N.

It follows that $0 \le g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \ge N$.

This proves the theorem.

Definition: Supremum norm

If X is a metric space, $\mathcal{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

Each $f \in C(x)$ its supremum norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded,

$$\|f\| < \infty.$$

It is obvious that $\|f\| = 0$ only if f(x)=0 for every $x \in X$, that is only if f=0.

If h=f+g, then

$$|h(x)| \le |f(x)| + |g(x)| \le \|f\| + \|g\|$$

For all $x \in X$;

Hence $\|f+g\| \le \|f\| + \|g\|$.

If we define the distance between $f \in C(x)$ and $g \in C(x)$ to be $\|f-g\|$,

It follows that for a metric are satisfied.

We have thus made $C(x)$ into a metric space.

A sequence $\{f_n\}$ converges to f with respect to the metric of $C(x)$ if and only if $f_n \to f$ uniformly on X.

Accordingly, closed subsets of $C(x)$ are sometimes called uniformly closed, the closure of a set $A \subset C(x)$ is called its uniform closure, and so on.

Theorem 2.7:

The above metric makes C(x) into a complete metric space.

Proof:

Let $\{f_n\}$ be a Cauchy sequence in C(x).

Each ε>0 corresponds as N such that

$$\|f_n - f_m\| < \varepsilon \text{ if } n \geq N \text{ and } m \geq N.$$

It follows by theorem 2.1, f is continuous.

Moreover, f is bounded, since there is an n such that

$$|f(x) - f_n(x)| < 1 \text{ for all } x \in X \text{ and } f_n \text{ is bounded.}$$

Thus $f \in C(x)$, and since $f_n \to f$ uniformly on X,

We have $\|f(x) - f_n(x)\| \to 0$ as $n \to \infty$.

UNIFORM CONVERGENCE AND INTEGRATION

Theorem 2.8:

Let α be monotonically increasing on [a, b]. Suppose $f_n \in \Re(\alpha)$ on [a, b], for n=1,2,3,…, and suppose $f_n \to f$ uniformly on [a, b]. Then $f \in \Re(\alpha)$ on [a, b], and

$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha \quad \text{-------(1)}$$

Proof:

It suffices to prove this for real f_n, Put $\varepsilon_n = \sup|f_n(x) - f(x)|$ ----------(2)

The supremum being taken over $a \le x \le b$.

Then $f_n - \varepsilon_n \le f \le f_n + \varepsilon_n$,

So that the upper and lower integrals of f satisfy

$$\int_a^b (f_n - \varepsilon_n) d\alpha \le \underline{\int} f d\alpha \le \overline{\int} f d\alpha \le \int_a^b (f_n + \varepsilon_n) d\alpha \text{ -------(3)}$$

Hence

$$0 \le \overline{\int} f d\alpha - \underline{\int} f d\alpha \le 2\varepsilon_n [\alpha(b) - \alpha(a)]$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ (Theorem 2.2), the upper and lower integrals are equal.

Thus $f \in \Re(\alpha)$.

Another application of (3) is

$$\left| \int_a^b f d\alpha - \int_a^b f_n d\alpha \right| \le \varepsilon_n [\alpha(b) - \alpha(a)] \text{ ------(4)}$$

This implies (1).

Corollary:

If $f_n \in \Re(\alpha)$ on [a, b] and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad (a \le x \le b),$$

The series converging uniformly on [a, b], then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$$ the series may be integrated term by term.

UNIFORM CONVERGENCE AND DIFFERENTIATION

Some stronger hypotheses are required for the assertion that $f_n' \to f'$ if $f_n \to f$.

Theorem 2.9:

Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [a, b]. If $\{f_n'\}$ converges uniformly on [a, b], then $\{f_n\}$ converges uniformly on [a, b], to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x), \quad (a \le x \le b). \quad \text{---------} (1)$$

Proof:

Let ε>0 be given.

Choose N such that $n \ge N, m \ge N$, implies

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad \text{-------} (2)$$

And $$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}, \quad (a \le t \le b) \quad \text{-------}(3)$$

If we apply the mean value theorem to the function f_n-f_m, (3) shows that

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \le \frac{|x-t|\varepsilon}{2(b-a)} \le \frac{\varepsilon}{2} \quad \text{-----} (4)$$

For any x and t on [a, b], if $n \ge N, m \ge N$. The equality

$$|f_n(x) - f_m(x)| \le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

Implies, by (2) & (4) that

$$|f_n(x) - f_m(x)| < \varepsilon, \quad (a \le x \le b, n \ge N, m \ge N),$$

So that $\{f_n\}$ converges uniformly on [a, b].

Let $f(x) = \lim_{n\to\infty} f_n(x), \quad (a \le x \le b)$

Fix a point x on [a, b] and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \phi(t) = \frac{f(t) - f(x)}{t-x} \quad \text{--------}(5)$$

For $a \le t \le b, t \ne x$.

Then $\lim_{t\to x}\phi_n(t) = f_n^{'}(x)$, (n=1,2,3,...) -------- (6)

The first inequality in (4) shows that

$$|\phi_n(t)-\phi_m(t)| \le \frac{\varepsilon}{2(b-a)}, \quad (n \ge N, m \ge N),$$

So that $\{\phi_n\}$ converges uniformly, for $t \ne x$.

If we now apply theorem 2.4 to $\{\phi_n\}$, (5) & (6) show that

$$\lim_{t\to x}\phi(t) = \lim_{n\to\infty} f_n^{'}(x);$$

This is the required one.

Theorem 2.10:

There exists a real continuous function on the real line which nowhere differentiable.

Proof:

Define $\varphi(x) = |x|$, $(-1 \le x \le 1)$ ------ (1)

And extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x+2) = \varphi(x) \quad \text{------------- (2)}$$

Then, for all s and t,

$$|\varphi(s)-\varphi(t)| \le |s-t| \quad \text{--------- (3)}$$

In particular, φ is continuous on R^1.

Define

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \varphi(4^n x) \quad \text{-------- (4)}$$

Since $0 \le \varphi \le 1$, theorem 2.3 shows that the series (4) converges uniformly on R^1.

By theorem 2.5, f is continuous on R^1.

Now fix a real number x and a positive integer m.

Put $\delta_m = \pm \frac{1}{2} 4^{-m}$ -------- (5)

Where the sign is so chosen that no integer lies between $4^m x$ and $4^m \left(x+\delta_m\right)$.

This can be done, since $4^m \left|\delta_m\right| = \frac{1}{2}$.

Define $\gamma_n = \frac{4^n \left(x+\delta_m\right) - \varphi(4^n x)}{\delta_m}$ -------- (6)

When n>m, then $4^n \delta_m$, so that $\gamma_n = 0$, when $0 \le n \le m$,

(3) implies that $\left|\gamma_n\right| \le 4^n$.

Since $\left|\gamma_m\right| = 4^m$,

It concludes that $\left|\frac{f\left(x+\delta_m\right) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right|$

$$\ge 3^m - \sum_{n=0}^{m-1} 3^n$$

$$= \frac{1}{2}(3^m + 1)$$

As $m \to \infty,\ \delta_m \to 0.$

It follows that f is not differentiable at x.

EQUICONTINUOUS FAMILIES OF FUNCTIONS

We know that every bounded sequence of complex numbers contains a convergent subsequence, and something similar is true for sequences of functions.

Here we define two kinds of boundedness.

Uniformly bounded

Let $\left\{f_n\right\}$ be a sequence of functions defined on a set E.

We say that $\{f_n\}$ is pointwise bounded on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x), \quad (x \in E, \text{ n=1,2,3,....}).$$

We say that $\{f_n\}$ is uniformly bounded on E if there exists a number M such that

$$|f_n(x)| < M, \quad (x \in E, \text{ n=1,2,3,....}).$$

Now if $\{f_n\}$ is pointwise bounded on E and E_1 is countable subset of E, it is always possible to find a subsequence $\{f_{nk}\}$ such that $\{f_{nk}(x)\}$ converges for every $x \in E_1$.

This can be done by the diagonal process which is used in the proof of Theorem 2.11.

Even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E.

Example 1:

Let $f_n(x) = \sin nx, \ (0 \le x \le 2\pi, \text{ n=1,2,3,...})$.

Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$.

In that case we must have

$$\lim_{k\to\infty}(\sin n_k x - \sin n_{k+1} x) = 0, \ (0 \le x \le 2\pi);$$

Hence $$\lim_{k\to\infty}(\sin n_k x - \sin n_{k+1} x)^2 = 0, \ (0 \le x \le 2\pi) \text{ ------- (1)}$$

By Lebesgue's theorem concerning integration of boundedly convergent sequence.

(1) Implies,

$$\lim_{k\to\infty}\int_0^{2\pi}(\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad \text{---------- (2)}$$

But a simple calculation shows that

$$\int_0^{2\pi}(\sin n_k x - \sin n_{k+1} x)^2 = 2\pi$$

Which contradicts (2).

Example 2:

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2} \;,\; (0 \le x \le 1,\ n=1,2,3,...).$$

Then $|f_n(x)| \le 1$, so that $\{f_n\}$ is uniformly bounded on [0, 1].

Also $\lim_{n\to\infty} f_n(x) = 0 \;,\; (0 \le x \le 1).$

But $f_n\left(\frac{1}{n}\right) = 1,\ (n=1,2,3,...)$

So that no subsequence can converge uniformly on [0, 1].

The concept which is needed in this connection is that of equicontinuity; it is given in the following definition.

Definition: Equicontinuous

A family $\mathcal{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every ε>0 there exists a δ>0 such that

$$|f(x) - f(y)| < \varepsilon$$

Whenever d(x,y)<δ, $x \in E$, $y \in E$, and $f \in \mathcal{F}$.

Here d denotes the metric of X.

It is clear that every member of an equicontinuous family is uniformly continuous.

The sequence given in Ex:2 is not equicontinuous.

Theorem 2.11:

If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{nk}\}$ such that $\{f_{nk}(x)\}$ converges for every $x \in E$.

Proof:

Let $\{x_i\}, i = 1, 2, 3, ...,$ be the points of E, arranged in a sequence.

Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence, which we shall denote by $\{f_{1,k}\}$ such that $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Now consider sequences $S_1, S_2, S_3, \ldots,$

We represent by the array

$$\begin{array}{l} S_1 : f_{1,1}\ f_{1,2}\ f_{1,3}\ f_{1,4}\ \ldots \\ S_2 : f_{2,1}\ f_{2,2}\ f_{2,3}\ f_{2,4}\ \ldots \\ S_3 : f_{3,1}\ f_{3,2}\ f_{3,3}\ f_{3,4}\ \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Which have the following properties:

(a) S_n is a sequence of S_{n-1}, for n=2,3,4,….
(b) $\{f_{n,k}(x_n)\}$ converges, as $k \to \infty$
(c) The order in which the functions appear is the same in each sequence:
(i.e.,) if one function precedes another in S_1, they are in the same relation in every S_n, until one or the other is deleted.
Hence when going from one row in the above array to the next below, functions may move to the left but never to the right.

Now we go down the diagonal of the array.

(i.e.,) we consider the sequence

$$S : f_{1,1}\ f_{2,2}\ f_{3,3}\ f_{4,4}\ \ldots,$$

By (c) the sequence S (except possibly its first n-1 terms) is a sub-sequence of S_n, for n=1,2,3,…

Hence (b) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \to \infty$, for every $x_i \in E$.

Theorem 2.12:

If K is compact metric space, if $f_n \in C(K)$ for n=1,2,3,…, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

Proof:

Let ε>0 be given.

Since $\{f_n\}$ converges uniformly, there is an integer N such that

$$\|f_n - f_N\| < \varepsilon,\ (n>N) \text{ --------- } (1)$$

Since continuous functions are uniformly continuous on compact sets, there is a δ>0 such that

$$|f_t(x) - f_t(y)| < \varepsilon \text{ ------------ } (2)$$

If $1 \le i \le N$ and $d(x, y) < \delta$.

If n>N and $d(x, y) < \delta$,it follows that

$$|f_n(x) - f_n(y)| \le |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon$$

In conjuction with (2), this proves the theorem.

Theorem 2.13:

If K is compact metric space, if $f_n \in C(K)$ for n=1,2,3,…, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{f_n\}$ is uniformly bounded on K.
(b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof:

(a) Let ε>0 be given.

Choose δ>0,

By the definition of equicontinuous,

$$|f_n(x) - f_n(y)| < \varepsilon \text{ ------------ } (1)$$

For all n, provided that $d(x, y) < \delta$.

Since K is compact, there are finitely many points $p_1, \dots, p_r$ in K

Such that to every $x \in K$ corresponds at least one p_i with $d(x, p_i) < \delta$.

Since $\{f_n\}$ is pointwise bounded,

There exist $M_i < \infty$ such that $|f_n(p_i)| < M_i$ for all n.

If M= max(M_1,…, M_r), then

$$|f_n(x)| < M + \varepsilon \text{ for every } x \in K.$$

Therefore $\{f_n\}$ is uniformly bounded on K.

(b) Let E be a countable dense subset of K.
Theorem 2.11. shows that $\{f_n\}$ has a subsequence $\{f_{ni}\}$ such that $\{f_{ni}(x)\}$ converges for every $x \in K$.
Put $f_{ni} = g_i$.
Now we have to prove that $\{g_i\}$ converges uniformly on K.
Let ε>0 be given.

Choose δ>0.

Let $V(x,\delta)$ be the set of all $y \in K$ with $d(x,y) < \delta$.

Since E is dense in K .

And K is compact,

There are finitely many points x_1,…, x_m in E such that

$$K \subset V(x_1,\delta) \cup ... \cup V(x_m,\delta) \text{ ------- (2)}$$

Since $\{g_i(x)\}$ converges for every $x \in K$, there is an integer N such that

$$|g_i(x_n) - g_j(x_n)| < \varepsilon \quad \text{---------- (3)}$$

Whenever $i \geq N,\ j \geq N,\ 1 \leq s \leq m$.

If $x \in K$, (2) shows that $x \in V(x_n, \delta)$ for some s,

So that $|g_i(x) - g_i(x_n)| < \varepsilon$

For everyi.

If $i \geq N,\ j \geq N,$ it follows from (3) that

$$|g_i(x) - g_j(x)| \leq |g_i(x) - g_j(x_n)| + |g_j(x_n) - g_j(x)| < 3\varepsilon$$

Hence $\{f_n\}$ contains a uniformly convergent subsequence.

THE STONE-WEIRSTRASS THEOREM

Theorem 2.14:

If f is a continuous complex function on [a, b], there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on [a, b]. If f is real, the P_n may be taken real.

Proof:

We assume that, without loss of generality, that [a, b]=[0, 1].

We may also assume that f(0)=f(1)=0.

For proving this theorem for this case,

Consider g(x)=f(x)-f(0)-x[f(1)-f(0)], $\left(0 \le x \le 1\right)$.

Here g(0)=g(1)=0,

and if g can be obtained as the limit of a uniformly convergent sequence of polynomials,

it is clear that the same is true for f,

since f-g is a polynomial.

We define f(x) to be zero for x outside [0, 1].

Then f is uniformly continuous on the whole line.

Put $Q_n(x) = c_n(1-x^2)^n, \quad \left(n = 1, 2, 3, \ldots\right)$ -------- (1)

Where c_n is chosen so that

$$\int_{-1}^{1} Q_n(x)dx = 1, \quad \left(n = 1, 2, 3, \ldots\right) \qquad \text{---------(2)}$$

We required some information about the order of magnitude of c_n.

Since $\int_{-1}^{1} (1-x^2)^n dx = 2\int_{0}^{1} (1-x^2)^n dx$

$$\geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-x^2)^n dx$$

$$\geq 2\int_0^{\frac{1}{\sqrt{n}}} (1-nx^2)dx$$

$$= \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$$

From (2), $c_n < \sqrt{n}$

The inequality $\left(1-x^2\right)^n \geq 1-nx^2$ which we used above is true by considering the function

$$\left(1-x^2\right)^n - 1 + nx^2$$

Which is zero at x=0 and whose derivative is positive in (0,1).

For any δ>0, (2) implies

$$Q_n(x) \leq \sqrt{n}(1-\delta^2)^n, \quad \left(\delta \leq |x| \leq 1\right), \quad \text{-------- (3)}$$

So that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$

Now set

$$P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt, \quad (0 \leq x \leq 1) \quad \text{--------- (4)}$$

The assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt,$$

$$= \int_0^1 f(t)Q_n(t-x)dt,$$

And the last integral is clearly a polynomial in x.

Thus $\{P_n\}$ is a sequence of polynomial, which are real if f is real.

Given $\varepsilon>0$, Choose $\delta>0$, such that $|y-x|<\delta$ implies

$$|f(y)-f(x)|<\frac{\varepsilon}{2}$$

Let M= sup|f(x)|.

Using (1) & (3),

And the fact that $Q_n(x)\geq 0$,

We see that for $0\leq x\leq 1$.

$$|P_n(x)\text{-}f(x)|=\left|\int_{-1}^{1}[f(x+t)-f(x)]Q_n(t)dt\right|$$

$$\leq\int_{-1}^{1}|f(x+t)-f(x)|Q_n(t)dt$$

$$\leq 2M\int_{-1}^{\delta}Q_n(t)dt+\frac{\varepsilon}{2}\int_{-\delta}^{\delta}Q_n(t)dt+2M\int_{\delta}^{1}Q_n(t)dt$$

$$\leq 4M\sqrt{n}(1\text{-}\delta^2)^n+\frac{\varepsilon}{2}$$

$$<\varepsilon$$

For all large enough n, which proves the theorem.

Corollary:

For every interval [-a, a] there is a sequence of real polynomials P_n such that $P_n(0)\to 0,$ as $n\to\infty$. The polynomials

$$P_n(x)=P_n^*(x)=P_n^*(0)\quad(n=1,2,3,\ldots)$$

Have desired properties.

Definition: An Algebra : A family 'A' of complex functions defined on a set E is said to be an algebra if

(a) $f+g\in A$

(b) $fg \in A$

(c) $cf \in A$, for all $f \in A, g \in A$ and for all complex constants c,

If A is closed under addition, multiplication and scalar multiplication.

If we consider algebras of real functions, in the case (iii) require for all real c.

If A has the property that $f \in A$, whenever $f_n \in A, (n = 1, 2, 3, ...)$ and $f_n \to f$ uniformly on E, then A is said to be **uniformly closed**.

Let B be the set of all functions which are limits of uniformly convergent sequences of members of A. Then B is called the **uniform closure of A.**

Example: The set of all polynomials is an algebra, and the Weiestrass theorem may be stated by saying that the set of continuous functions on [a, b] is the uniform closure of the set of polynomials on [a, b].

Theorem 2.15: Let B be the uniform closure of an algebra A of bounded functions. Then B is uniformly closed algebra.

Proof:

If $f \in B$, and $g \in B$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that

$$f_n \to f \ , g_n \to g \text{ and } f_n \in A \ , g_n \in A \ .$$

Since give function is bounded, it is easy to show that

$$f_n + g_n \to f + g, \ f_n g_n \to fg, \ cf_n \to cf \ ,$$

Where c is any constant, the convergence being uniform in each case.

Hence $f + g \in B, \ fg \in B, \ cf \in B$,

So that B is an algebra.

By theorem 2.14, B is (Uniformly) closed.

Definition: Separate points:

Let A be a family of functions on a set E. Then A is said to separate points on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in A$ such that $g(x) \neq 0$, it is say that **A vanishes at no point of E.**

Example: An example ofan algebra which does not separate points is the set of all even polynomials, say on [-1, 1], since f(-x)=f(x) for every even function f.

Theorem 2.16: Suppose A is an algebra of function on a set E, A separates point on E, and A vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1,c_2 are constants (real if A is a real algebra). Then A contains a function f such that

$$f(x_1) = c_1, \ \ f(x_2) = c_2.$$

Proof:

Let us assume A contains the functions g, h, k such that

$$g(x_1) \neq g(x_2), \ h(x_1) \neq 0, \ k(x_2) \neq 0$$

Put $\qquad u = gk - g(x_1)k, \ v = gh - g(x_2)h$

Then $\qquad u \in A, v \in A, u(x_1) = v(x_2) = 0, u(x_2) \neq 0, v(x_1) \neq 0$

Therefore $\qquad f = \dfrac{c_1 v}{v(x_1)} + \dfrac{c_2 u}{u(x_2)}$

Has the required properties.

Theorem 2.17: Let A be an algebra of real continuous functions on a compact set K. If A separates points on K and if A vanishes at no point of K, then the uniform closure B of A consists of all real continuous functions on K.

We prove this theorem into four steps:

STEP 1: If $f \in B$ then $|f| \in B$.

Proof: Let $a = \sup|f(x)|, \ (x \in K)$ -------- (1)

And let ε>0 be given.

By corollary 2.14, there exist real numbers $c_1, \ldots, c_n$ such that

$$\left|\sum_{i=1}^{n} c_i y^i - |y|\right| < \varepsilon, \ \ (-a \leq y \leq a) \quad \text{--------(2)}$$

Since B is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i \quad \text{is a member of B.}$$

By (1) & (2), we have

$$\left|g(x) - |f(x)|\right| < \varepsilon, \; (x \in K).$$

Since B is uniformly closed, this shows that $|f| \in B$.

STEP 2: If $f \in B$ and $g \in B$, then

By max(f, g) we mean the function h is defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$$

In the same way we define min(f, g).

Proof:

It follows from the step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2},$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, ..., f_n \in B,$ then $\max(f_1, ..., f_n) \in B$ and $\min(f_1, ..., f_n) \in B$

STEP 3: Given a real function f, continuous on K, a point $x \in K,$ and $\varepsilon > 0,$ there exists a function $g_x \in B$ such that $g_x(x) = f(x)$ and

$$g_x(t) > f(t) - \varepsilon, \quad (t \in K) \text{ ------- (3)}$$

Proof: since $A \subset B$ and A satisfies the hypotheses of theorem 2.16 so does B.

Hence for every $y \in K$, we can find a function $h_y \in B$ such that

$$h_y(x) = f(x), \quad h_y(y) = f(y) \text{ ------- (4)}$$

by the continuity of h_y there exists an open set J_y, containing y, such that

$$h_y(t) > f(t) - \varepsilon,\ (t \in J_y) \quad \text{------}(5)$$

Since K is compact, there is finite set of points $y_1, ..., y_n$ such that

$$K \subset J_{y_1} \cup ... \cup J_{y_n} \quad \text{---------}(6)$$

Put $$g_x = \max\left(h_{y_1}, ..., h_{y_n}\right).$$

By step2, $g_x \in B$, and the relations (4) to (6) show that g_x has the other required properties.

STEP 4: Given a real function f, continuous on K, and ε>0, there exists a function $h \in B$ such that

$$\left|h(x) - f(x)\right| < \varepsilon,\ (x \in K) \quad \text{-------}\ (7)$$

Since B is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof:

Let us consider the function g_x, for each $x \in K$, constructed in step 3.

By the continuity of g_x, there exist open sets V_x containing x, such that

$$g_x(t) < f(t) + \varepsilon,\ (t \in V_x) \quad \text{--------}\ (8)$$

Since K is compact, there exists a finite set of points $x_1, ..., x_m$ such that

$$K \subset V_{x_1} \cup ... \cup V_{x_n} \quad \text{---------}(9)$$

Put $$h = \min\left(g_{x_1}, ..., g_{x_m}\right).$$

By step 2, $h \in B$ and (3) implies

$$h(t) > f(t) - \varepsilon,\ (t \in K) \quad \text{-------}\ (10)$$

Whereas (8) & (9) imply

$$h(t) < f(t) + \varepsilon,\ (t \in K) \quad \text{------}\ (11)$$

Finally , (7) follows from (10) & (11).

Definition : Self-adjoint:

A is said to be self-adjoint , if for every $f \in A$ its complex conjugate $\bar{f}$ must also belongs to A. $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

UNIT-III

Measure on the Real line

R - the real line

interval $I = [a, b)$, where a & b are finite,

when $a = b$, $I = \phi$

$l(I)$ - length of I

$= b - a$

The Lebesgue outer measure

The outer measure of a set is given by $m^*(A) = \inf \sum l(I_n)$, where the infimum is taken over all finite or countable collections of intervals $[I_n]$ such that

$$A \subseteq \cup I_n.$$

Theorem 3.1.1:

(i) $m^*(A) \geq 0$,

(ii) $m^*(\phi) = 0$,

(iii) $m^*(A) \leq m^*(B)$ if $A \subseteq B$,

(iv) $m^*([x]) = 0$ for any $x \in R$.

Proof: Proof (i), (ii) & (iii) are obvious.

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Since $x + I_n = [x, x+(1/n))$ for each n,

& $l(I_n) = 1/n$.

(iv) is also obvious.

Example 1: Show that for any set A,

$m^*(A) = m^*(A+x)$, where $A+x = [y+x : y \in A]$,

(ie) outer measure is translation invariant.

Soln: For each $\epsilon > 0$,

there exists a collection $[I_n]$ such that

$$A \subseteq \cup I_n$$

and $m^*(A) \geq \Sigma l(I_n) - \epsilon$.

but $A+x \subseteq \cup (I_n + x)$.

So, for each ϵ,

$$m^*(A+x) \leq \Sigma l(I_n+x) = \Sigma l(I_n)$$

$$\leq m^*(A) + \epsilon.$$

So $m^*(A+x) \leq m^*(A)$.

But $A = (A+x) - x$

So we have $m^*(A) \leq m^*(A+x)$.

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Theorem 3.1.2:

The outer measure of an interval equals its length.

Proof:

Case (i): If I is a closed interval $[a,b]$.

Then for each $\epsilon > 0$,

from the defn. of outer measure and theorem 3.1.1,

$$m^*([a,b]) \le m^*([a, b+\epsilon))$$
$$\le b-a+\epsilon,$$

So $m^*(I) \le b-a$. $\longrightarrow$ (1)

To obtain the opposite inequality:

for each $\epsilon > 0$,

I may be covered by a collection of intervals $[I_n]$ such that

$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon,$$

where $I_n = [a_n, b_n)$ say.

For each n,

let $I_n' = (a_n - \epsilon/2^n, b_n)$

then $\bigcup_{n=1}^{\infty} I_n' \supseteq I$,

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So by the Heine - Borel theorem,

a finite subcollection of I_n',

say $J_1, \ldots, J_N$ where $J_K = (c_k, d_k)$,

covers I.

Then, as we may suppose that no J_k is contained in any other

we have, supposing that $c_1 < c_2 < \ldots < c_N$,

$$d_N - c_1 = \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} (d_k - c_{k+1})$$

$$< \sum_{k=1}^{N} l(J_k).$$

So we have

$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon \geq \sum_{n=1}^{\infty} l(I_n') - 2\epsilon$$

$$\geq \sum_{k=1}^{N} l(J_k) - 2\epsilon$$

$$\geq d_N - c_1 - 2\epsilon$$

$$> b - a - 2\epsilon$$

$$> l(I) - 2\epsilon. \longrightarrow (2)$$

Then from (1) & (2), it is proved.

<u>Case (ii) :</u>

If $I = (a, b]$,

and $a > -\infty$.

If $a = b$, theorem 3.1.1 gives the result.

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If $a<b$,

suppose that $0<\epsilon<b-a$

and write $I' = [a+\epsilon, b]$.

Then $m^*(I) \geq m^*(I') = l(I) - \epsilon \rightarrow (3)$

But $I \subseteq I'' = [a, b+\epsilon)$,

so $m^*(I) \leq l(I'') = l(I) + \epsilon. \rightarrow (4)$

Since (3) & (4) are true for all small ϵ,

$$m^*(I) = l(I).$$

|||ly we can consider the cases

$I = (a,b)$ & $I = [a,b)$.

Case (iii) :

suppose that I is an infinite interval.

Four type of interval occur.

suppose that $I = (-\infty, a]$, other 3 cases are similar.

For any $M>0$,

there exists k such that the finite

interval I_M,

where $I_M = [k, k+M]$, is contained in I.

so $m^*(I) > M$

and hence $m^*(I) = \infty = l(I)$.

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Theorem 3.1.3:

[This theorem asserts that m^* has the property of countable subadditivity.]

For any sequence of sets $\{E_i\}$,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$

Proof:

For each i,

& for any $\epsilon > 0$,

there exists a sequence of intervals $\{I_{i,j}, j = 1, 2, \dots\}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$

and $$m^*(E_i) \geq \sum_{j=1}^{\infty} l(I_{i,j}) - \epsilon/2^i.$$

Then $$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j},$$

(ie). the sets $[I_{i,j}]$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$.

So $$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i,j=1}^{\infty} l(I_{i,j}) \leq \sum_{i=1}^{\infty} m^*(E_i) + \epsilon.$$

ϵ is arbitrary.

Hence proved.

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Example 2: Show that, for any set A and any $\epsilon > 0$, there is an open set O containing A and such that

$$m^*(O) \leq m^*(A) + \epsilon.$$

Solution:

Choose a sequence of intervals I_n such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$

and $\sum_{n=1}^{\infty} l(I_n) - \epsilon/2 \leq m^*(A)$.

If $I_n = [a_n, b_n)$,

let $I_n' = (a_n - \epsilon/2^{n+1}, b_n)$

so that $A \subseteq \bigcup_{n=1}^{\infty} I_n'$.

Hence if $O = \bigcup_{n=1}^{\infty} I_n'$, O is an open set

and
$$m^*(O) \leq \sum_{n=1}^{\infty} l(I_n')$$
$$= \sum_{n=1}^{\infty} l(I_n) + \epsilon/2$$
$$\leq m^*(A) + \epsilon.$$

Example 3:

Suppose that in the definition of outer measure, $m^*(E) = \inf \sum l(I_n)$ for sets $E \subseteq \mathbb{R}$, we require

(i) I_n open, (ii) $I_n = [a_n, b_n)$,

(iii) $I_n = (a_n, b_n]$, (iv) I_n closed (or)

(v) mixtures are allowed, for different n, of the various types of interval.

Show that the same m^* is obtained.

Solution:

In case (ii) we obtain the m^* of definition of outer measure.

we write m^* as m_o^* in Case (i),

m_{oc}^* in Case (iii),

m_c^* in Case (iv),

m_m^* in Case (v).

we have to show that each equals m_m^*.

Consider m_o^*, Simillarly we prove the other cases.

From the definition,

$$m_m^*(E) \leq m_o^*(E).$$

To prove the Converse:

for each $\epsilon > 0$,

and each interval I_n, let I_n' be an open interval containing I_n with

$$l(I_n') = (1+\epsilon)\, l(I_n).$$

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Suppose that the sequence $\{I_n\}$ is such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n$$

and $m_m^*(E) \le \sum_{n=1}^{\infty} l(I_n) - \epsilon$.

Then $m_m^*(E) + \epsilon \geq (1+\epsilon)^{-1} \sum_{n=1}^{\infty} l(I_n')$.

But $E \subseteq \bigcup_{n=1}^{\infty} I_n'$, a Union of open intervals,

So $m_o^*(E) \le (1+\epsilon) m_m^*(E) + \epsilon(1+\epsilon)$,

for any $\epsilon > 0$, so $m_o^*(E) \le m_m^*(E)$.

as we required.

Measurable sets

Lebesgue measurable: The set E is Lebesgue measurable or measurable if for each set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap CE).$$

As m^* is subadditive, to prove E is measurable we need only show, for each A, that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap CE).$$

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Example 4:

Show that if $m^*(E)=0$ then E is measurable.

Soln: By Theorem 1 (iii),

$$m^*(A) \geq m^*(A\cap E) + m^*(A\cap CE)$$ is

satisfied for each A.

Definition : σ- algebra :

A class of subsets of an arbitrary space X is said to be a σ-algebra or a σ-field, if X belongs to the class and the class is closed under the formation of countable unions and of complements.

Algebra :

In the definition of σ-algebra, if we consider only finite unions, we get an algebra or a field.

$\mathcal{M}$ - the class of Lebesgue measurable sets.

Theorem 3.1.4:

The class $\mathcal{M}$ is a σ-algebra.

Proof: From the definition of Lebesgue measurable, $R \in \mathcal{M}$ and the symmetry in that definition between E and CE implies that

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if $E \in \mathcal{M}$ then $CE \in \mathcal{M}$.

So it remains to be shown that

if $\{E_j\}$ is a sequence of sets in $\mathcal{M}$

then $E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$.

Let A be an arbitrary set.

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap CE_1)$$

and again,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2 \cap CE_1) + m^*(A \cap CE_1 \cap CE_2)$$

Continuing in this way we obtain, for $n \geq 2$,

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap \bigcap_{j<i} CE_j) + m^*(A \cap \bigcap_{j=1}^{n} CE_j)$$

$$= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j) + m^*(A \cap C \bigcup_{j=1}^{n} E_j)$$

$$\geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j) + m^*(A \cap C \bigcup_{j=1}^{\infty} E_j),$$

by theorem 3.1.1 (iii),

$$\therefore m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*(A \cap E_i \cap c \bigcup_{j<i} E_j)$$

$$+ m^*(A \cap c \bigcup_{j=1}^{\infty} E_j)$$

$$\geq m^*(A \cap \bigcup_{j=1}^{\infty} E_j) + m^*(A \cap c \bigcup_{j=1}^{\infty} E_j)$$

$$\geq m^*(A). \longrightarrow (1)$$

Using the theorem 3.1.3 twice,

for any n,

$$\bigcup_{i=1}^{n} (E_i \cap c \bigcup_{j<i} E_j) = \bigcup_{i=1}^{n} E_i .$$

Hence we have equality throughout in (1),

It is shown that

$\bigcup_{j=1}^{\infty} E_j$ is measurable.

Example 5:

Show that if $F \in \mathcal{M}$ and $m^*(F \Delta G) = 0$, then G is measurable.

Soln: By Example 4,

$F \Delta G$ ~~are~~ is measurable.

So its subsets $F-G$ and $G-F$ are measurable.

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by theorem 3.1.4,

$F \cap G = F - (F - G)$ is measurable.

So $G = (F \cap G) \cup (G - F)$ is measurable.

Theorem 3.1.5:

If $\{E_i\}$ is any sequence of disjoint measurable sets then $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i) \rightarrow (1)$

(ie) m^* is countably additive on disjoint sets of $\mathcal{M}$.

Proof:

Let $A = \bigcup_{i=1}^{\infty} E_i$ in theorem 3.1.4,

which we have seen to be an equality,

that expression simplifies, since the sets E_i are disjoint.

If we put $E_i = \phi$ in (1),

the same result for finite unions follows as a special case.

If E is a measurable set,

we write $m(E) = m^*(E)$.

Then the set function m is defined on the σ-algebra $\mathcal{M}$ of measurable sets.

This theorem states that m is countably additive set function, and $m(E)$ is called the Lebesgue measure of E.

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Theorem 3.1.6:

Every interval is measurable.

Proof:

If the interval to be of the form $[a, \infty)$, as theorem 3.1.4, then it gives the result for the other types of interval.

For any set A,

we show that

$$m^*(A) \geq m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)). \rightarrow (1)$$

write $A_1 = A \cap (-\infty, a)$

and $A_2 = A \cap [a, \infty)$.

Then for any $\epsilon > 0$ there exist intervals I_n

such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$

and $m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon$.

write $I_n' = I_n \cap (-\infty, a)$

and $I_n'' = I_n \cap [a, \infty)$,

so that $l(I_n) = l(I_n') + l(I_n'')$.

Then $A_1 \subseteq \bigcup_{n=1}^{\infty} I_n'$,

$A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''$

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So $m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n') + \sum_{n=1}^{\infty} l(I_n'')$ (15)

$$\leq \sum_{n=1}^{\infty} l(I_n) \leq m^*(A) + \epsilon.$$

$\therefore$ Every interval is measurable.

Theorem 3.1.7:

Let $\mathcal{A}$ be a class of subsets of a space X. Then there exists a smallest σ-algebra $\mathcal{S}$ containing $\mathcal{A}$. We say that $\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$.

Proof: Let $[\mathcal{S}_\alpha]$ be any collection of σ-algebras of subsets of X.

Then from the definition of σ-algebra,

$\bigcap_\alpha \mathcal{S}_\alpha$ is a σ-algebra.

But there exists a σ-algebra containing $\mathcal{A}$, namely the class of all subsets of X.

So taking the intersection of the σ-algebras containing $\mathcal{A}$ we get the σ-algebra, necessarily the smallest, containing $\mathcal{A}$.

If we replace σ-algebra by algebra, the class obtained being the generated algebra.

The proof is the same.

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Definition : Borel sets :

The members of the σ- algebra generated by the class of intervals of the form $[a,b)$ are called the Borel sets of R.

It is denoted by $\mathcal{B}$.

Theorem 3.1.8:

(i) $\mathcal{B} \subseteq \mathcal{M}$, (ie) every Borel set is measurable.

(ii) $\mathcal{B}$ is the σ- algebra generated by each of the following classes:

the open intervals, the open sets, the G_δ-sets, the F_σ - sets.

Proof:

(i) from theorem 3.1.4 & 3.1.6, it is proved.

(ii) Let $\mathcal{B}_1$ be the σ- algebra generated by the open intervals.

Every open interval is a Borel set.

[$\because$ it is the union of a sequence of intervals of the form $[a,b)$.]

So $\mathcal{B}_1 \subseteq \mathcal{B}$.

But every interval $[a,b)$, is the intersection of a sequence of open intervals

and so $\mathcal{B} \subseteq \mathcal{B}_1$.

$\therefore$ $\mathcal{B} = \mathcal{B}_1$.

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Since every open set is the union of a sequence of open intervals the second result follows.

(iii) Since G_δ - sets and F_σ - sets are formed from open sets using only countable intersections and complements the result in these cases.

<u>Example 6:</u> For any set A there exists a measurable set E containing A and such that

$$m^*(A) = m(E).$$

<u>Soln:</u> In Example 2,

let $\epsilon = 1/n$ and O_n - open set.

Then the G_δ-set $E = \bigcap_{n=1}^{\infty} O_n$ has the required properties,

Since for every n,

$$m(E) \leq m(O_n) \leq m^*(A) + 1/n.$$

<u>Definition:</u>

For any sequence of sets $\{E_i\}$

$$\limsup E_i = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} E_i, \quad \liminf E_i = \bigcup_{n=1}^{\infty} \bigcap_{i \geq n} E_i.$$

It is from $\liminf E_i \subseteq \limsup E_i$.

If both are equal, it is denoted by $\lim E_i$.

$\limsup E_i$ is the set of points belonging to infinitely many of the sets E_i.

$\liminf E_i$ is the set of points belonging to all but finitely many of the sets E_i.

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It is also immediate that if $E_1 \subseteq E_2 \subseteq \ldots$,

then $\lim E_i = \bigcup_{i=1}^{\infty}$,

and that if $E_1 \supseteq E_2 \supseteq \ldots$,

Then $\lim E_i = \bigcap_{i=1}^{\infty} E_i$.

Theorem 3.1.9: Let $\{E_i\}$ be sequence of measurable sets. Then (i) if $E_1 \subseteq E_2 \subseteq \ldots$, then $m(\lim E_i) = \lim m(E_i)$.

(ii) if $E_1 \supseteq E_2 \supseteq \ldots$, and $m(E_i) < \infty$ for each i, then $m(\lim E_i) = \lim m(E_i)$.

Proof: (i) Write $F_1 = E_1$, $F_i = E_i - E_{i-1}$ for $i > 1$.

Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and the sets F_i are measurable and disjoint.

So $m(\lim E_n) = m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(F_i) = \lim \sum_{i=1}^{n} m(F_i)$

$= \lim m\left(\bigcup_{i=1}^{n} F_i\right)$

$= \lim m(E_n)$.

(ii) Given $E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \ldots$

by (i) $m(\lim (E_1 - E_i)) = \lim (m(E_1 - E_i))$

$= m(E_1) - \lim m(E_i)$.

but $\lim (E_1 - E_i) = \bigcup_{i=1}^{\infty} (E_1 - E_i)$

$= E_1 - \bigcap_{i=1}^{\infty} E_i = E_1 - \lim E_i$

Taking measures of both sides we get the result. [$\because m(E_1) < \infty$]

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Regularity

A regular measure:

A non-negative countably additive set function satisfying the conditions

(i) $\forall \epsilon > 0$, $\exists O$, an open set, $O \supseteq E$ s.t. $m^*(O-E) \leq \epsilon$

(ii) $\exists F$, an F_σ-set, $F \subseteq E$ s.t. $m^*(E-F) = 0$

is said to be a regular measure.

Theorem 3.1.10

The following statements regarding the set E are equivalent:

(i) E is measurable,

(ii) $\forall \epsilon > 0$, $\exists O$, an open set, $O \supseteq E$ s.t. $m^*(O-E) \leq \epsilon$,

(iii) $\exists G$, a G_δ-set, $G \supseteq E$ s.t. $m^*(G-E) = 0$,

(ii)* $\forall \epsilon > 0$, $\exists F$, a closed set, $F \subseteq E$ s.t. $m^*(E-F) \leq \epsilon$,

(iii)* $\exists F$, an F_σ-set, $F \subseteq E$ s.t. $m^*(E-F) = 0$.

Proof: (i) $\Rightarrow$ (ii):

Given E is measurable.

So $m(E) < \infty$.

As in Ex-2, there is an open set $O \supseteq E$

such that $m(O) \leq m(E) + \epsilon$.

So $m(O-E) = m(O) - m(E) \leq \epsilon$.

If $m(E) = \infty$,

Take $R = \bigcup_{n=1}^{\infty} I_n$, a union of disjoint finite intervals.

Then if $E_n = E \cap I_n$,

we have $m(E_n) < \infty$

So there is an open set $O_n \supseteq E_n$

such that $m(O_n - E_n) \leq \epsilon / 2^n$.

Take $O = \bigcup_{n=1}^{\infty} O_n$, an open set.

Then $O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (O_n - E_n)$.

So $m(O-E) \leq \sum_{n=1}^{\infty} m(O_n - E_n) \leq \epsilon$.

(ii) ⇒ (iii):

for each n,

let O_n be an open set,

$O_n \supseteq E$, $m^*(O_n - E) < 1/n$.

If $G = \bigcap_{n=1}^{\infty} O_n$,

G is a G_δ-set,

$E \subseteq G$ and $m^*(G-E) \leq m^*(O_n - E) < 1/n$.

for each n.

The result proved.

(iii) ⇒ (i): $E = G - (G - E)$, the set G is measurable.

by Example 4, $G - E$ is measurable.

$\therefore$ E is measurable.

(i) ⇒ (ii)*: CE is measurable,

there exist an open set O,

such that $O \supseteq CE$ and $m(O - CE) \leq \epsilon$.

but $O - CE = E - CO$

If we put $F = CO$, it is proved.

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(ii)* ⇒ (iii)*:

for each n,

let F_n be a closed set,

$F_n \subseteq E$

and $m^*(E-F_n) < 1/n$.

If $F = \bigcup_{n=1}^{\infty} F_n$, F is an F_σ-set,

$F \subseteq E$, for each n,

$m^*(E-F) \leq m^*(E-F_n) < 1/n$.

Hence proved.

(iii)* ⇒ (i): Since $E = F \cup (E-F)$

we prove E is measurable.

Theorem 3.1.11:

If $m^*(E) < \infty$ then E is measurable if and only if, $\forall\ \epsilon > 0$, $\exists$ disjoint finite intervals $I_1, \ldots, I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$. We may require that the intervals I_i be open, closed or half-open.

Proof:

Given E is measurable.

Then by previous theorem,

$\forall\ \epsilon > 0$, $\exists$ an open set O containing E with

$m(O-E) < \epsilon$.

As $m(E)$ is finite then $m(O)$ is finite.

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by theorem 3.1.9,

O is the union of disjoint open intervals

$I_i, i=1,2,\ldots$

so by theorem 3.1.5,

$\exists n$ such that $\sum_{i=n+1}^{\infty} l(I_i) < \epsilon$.

Take $U = \bigcup_{i=1}^{n} I_i$.

Then $E \Delta U = (E-U) \cup (U-E) \subseteq (O-U) \cup (O-E)$

$\therefore m^*(E \Delta U) < 2\epsilon$.

If we take the half-open interval,

we first obtain open intervals $I_1, \ldots, I_n$ as above,

then for each i,

choose a half-open interval $J_i \subset I_i$

such that $m(I_i - J_i) < \epsilon/n$.

Then the intervals J_i are disjoint,

by ex-1,

$m(E \Delta \bigcup_{i=1}^{n} J_i) \leq m(E \Delta \bigcup_{i=1}^{n} I_i) + m(\bigcup_{i=1}^{n} J_i \Delta \bigcup_{i=1}^{n} J_i) < 2\epsilon$,

so it is for the intervals J_i.

<u>Conversely:</u>

by Example 2,

$\forall \epsilon > 0$, $\exists$ O open, $O \supseteq E$

such that $m(O) \leq m^*(E) + \epsilon$.

If we can show that $m^*(O-E)$ can be made arbitrarily small,

then E is measurable. [$\because$ last theorem]

write $J = \bigcup_{i=1}^{n} I_i$

& $U = O \cap J$.

Then by Ex-1, and Subadditivity

$m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E)$.

Since $U \subseteq J$,

we have $U - E \subseteq J - E$

and since $E \subseteq O$,

we have $E - U = E - J$.

so $U \Delta E \subseteq J \Delta E$

$m^*(U \Delta E) < \epsilon$.

but $E \subseteq U \cup (U \Delta E)$,

so $m^*(E) \leq m(U) + \epsilon$.

so by $m(O) \leq m^*(E) + \epsilon$,

we have $m(O \Delta U) = m(O - U) = m(O) - m(U)$

$\leq m^*(E) - m(U) + \epsilon < 2\epsilon$.

Then by $m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E)$,

we have $m^*(O - E) = m^*(O \Delta E) < 3\epsilon$.

as required.

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Measurable Functions:

The extended real-number system, if we add ∞ and $-\infty$ to the real number system with the conventions that

$$a + \infty = \infty \quad (a \text{ real})$$

$$a \cdot \infty = \infty \quad (a > 0)$$

$$a \cdot \infty = -\infty \quad (a < 0), \quad \infty \cdot \infty = \infty, \quad 0 \cdot \infty = 0.$$

$|||^{ly}$ we can have for $-\infty$.

Lebesgue-measurable function: Let f be an extended real-value function defined on a measurable set E. Then f is a Lebesgue-measurable function or a measurable function if, for each $\alpha \in R$, the set $[x : f(x) > \alpha]$ is measurable.

The domain of definition of f will be either R or $R-F$ where $m(F)=0$.

Example: Show that if f is measurable, then $[x : f(x) = \alpha]$ is measurable for each extended real number α.

Solution: For finite α,

$$[x : f(x) = \alpha] = [x : f(x) \geq \alpha] \cap [x : f(x) \leq \alpha]$$

and so is measurable.

For $\alpha = \infty$:

$$[x : f(x) = \infty] = \bigcap_{n=1}^{\infty} [x : f(x) \geq n],$$

a measurable set.

$|||^{ly}$ it proved for $\alpha = -\infty$.

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Example: The constant functions are measurable.

Soln: Depending on the choice of α,

the set $[x : f(x) > \alpha]$,

where f is constant, is the whole real line or the empty set.

* The characteristic function χ_A of the set A, is measurable iff A is measurable.

* Continuous functions are measurable.

Borel measurable or a Borel function:

We say that the function f is Borel measurable or a Borel function if

$\forall \alpha$, $[x : f(x) > \alpha]$ is a Borel set.

Almost everywhere [a.e]

If a property holds except on a set of measure zero, we say that it holds almost everywhere. [a.e].

Note:

(i) If f be a measurable function and if $f = g$ a.e. Then g is measurable.

(ii) Let $\{f_i\}$ be a sequence of measurable functions converging a.e to f; then f is measurable, since $f = \lim\sup f_i$, a.e.

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(iii) If f is a measurable fn., then

$$f^+ = \max(f, 0) \ \& \ f^- = -\min(f, 0)$$

(iv) The set of points on which sequence of measurable functions $\{f_n\}$ Converges, is measurable.

<u>Essential supremum:</u>

Let f be a measurable function : then $\inf [\alpha : f \leq \alpha \text{ a.e}]$ is called the essential supremum of f, denoted by ess sup f.

<u>Essential infimum:</u>

Let f be a measurable function ; Then $\sup [\alpha : f \geq \alpha \text{ a.e}]$ is called the essential infimum of f, denoted by ess inf f.

<u>Essentially bounded:</u>

If f is a measurable function and ess sup $|f| < \infty$, then f is said to be essentially bounded.

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Integration of Functions of a Real Variable

A non-negative finite-valued function $\varphi(x)$, taking only a finite number of different value, is called a simple function.

If $a_1, a_2, \ldots, a_n$ are the distinict values taken by φ and $A_i = \{x : \varphi(x) = a_i\}$, then

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x). \rightarrow ①$$

The set A_i are measurable if φ is a measurable function.

Integral of φ.

Let φ be a measurable simple function.

Then $\int \varphi \, dx = \sum_{i=1}^{n} a_i m(A_i)$,

where $a_i, A_i, i = 1, \ldots, n$ are as in ①, is called the integral of φ.

Example 1: Let the sets A_i be defined above.

Then $A_i \cap A_j = \phi, \quad i \neq j$

and $\bigcup_{i=1}^{n} A_i = R$.

Definition:

For any non-negative measurable function f, the integral of f,

$\int f \, dx = \sup \int \varphi \, dx$, where the supremum is taken over all measurable simple fns. φ, $\varphi \leq f$.

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Integral of f over E:

For any measurable set E, and any non-negative measurable function f,

$$\int_E f\,dx = \int f \chi_E \,dx$$ is the integral of f over E.

Theorem 3.2.1: If φ is a measurable simple function, then in the above definitions,

(i) $\int_E \varphi\,dx = \sum_{i=1}^{n} a_i m(A_i \cap E)$ for any measurable set E,

(ii) $\int_{A\cup B} \varphi\,dx = \int_A \varphi\,dx + \int_B \varphi\,dx$ for any disjoint measurable sets A and B,

(iii) $\int a\varphi\,dx = a\int \varphi\,dx$ if $a>0$.

Proof:

(i) from the above definitions (i) is proved.

(ii) $$\int_A \varphi\,dx + \int_B \varphi\,dx = \sum_{i=1}^{n} a_i m(A\cap A_i) + \sum_{i=1}^{n} a_i m(B\cap A_i)$$

$$= \sum_{i=1}^{n} a_i m((A\cup B)\cap A_i)$$

$$= \int_{A\cup B} \varphi\,dx.$$

(iii) As φ takes the values a_i, $a\varphi$ takes the distinct values aa_i,

so $$\int a\varphi\,dx = \sum_{i=1}^{n} aa_i m(A_i) = a\int \varphi\,dx.$$

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Example 2: Show that if f is a non-negative measurable function, then $f=0$ a.e. if and only if

$$\int f\,dx = 0.$$

Soln: If $f=0$ a.e.

and φ is a measurable simple function,

$$\varphi \leq f,$$

Then clearly $\int \varphi\,dx = 0$,

from the definition, $\int f dx = 0$.

Conversely,

if $\int f dx = 0$

and $E_n = [x : f(x) \geq 1/n]$,

then $\int f dx \geq \int n^{-1} \chi_{E_n} dx = n^{-1} m(E_n)$.

So $m(E_n) = 0$.

But $[x : f(x) > 0] = \bigcup_{n=1}^{\infty} E_n$,

So $f = 0$ a.e.

Theorem 3.2.2:

Let f and g be the non-negative measurable functions.

(i) If $f \leq g$, then $\int f dx \leq \int g\,dx$.

(ii) If A is a measurable set and $f \leq g$ on A,

then $\int_A f dx \leq \int_A g\,dx$.

(iii) If $a \geq 0$, then $\int af\,dx = a\int f dx$.

(iv) If A and B are measurable sets & $A \supseteq B$ then

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$$\int_A f\,dx \geq \int_B f\,dx.$$

Proof:

(i) It is proved [∵ Definitions].

(ii) "   "

(iii) is obvious if $a=0$.

If $a>0$, φ is a measurable simple fn. with

$\varphi \leq af$ if and only if,

$$\varphi = a\psi$$

Where ψ is simple $\psi \leq f$ and

then $\int \varphi\,dx = a\int \psi\,dx$ [∵ Th: 3.2.1 (iii)]

$$\text{so } \int af\,dx = \sup \int \varphi\,dx$$

$$= a \sup \int \psi\,dx$$

$$= a\int f\,dx.$$

(iv) $\chi_A f \geq \chi_B f$ & applied in (i).

it is proved.

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Theorem 3.2.3: Fatou's Lemma

Let $\{f_n, n=1,2,\dots\}$ be a sequence of non-negative measurable functions.

Then $\liminf \int f_n\,dx \geq \int \liminf f_n\,dx$. $\rightarrow$ (1)

Proof:

Let $f = \liminf f_n$.

Then f is a non-negative measurable function.

for each measurable simple function φ with

$$\varphi \leq f,$$

we have $\int \varphi\,dx \leq \liminf \int f_n\,dx$. $\rightarrow$ (2)

Case (i): $\int \varphi\,dx = \infty$.

Then from definitions, for some measurable set A,

we have $m(A) = \infty$,

and $\varphi > a > 0$ on A.

write $g_k(x) = \inf_{j \geq k} f_j(x)$

and $A_n = [x : g_k(x) > a,\ \text{all } k \geq n]$,

a measurable set.

Then $A_n \subseteq A_{n+1}$, each n.

but, for each x, $\{g_k(x)\}$ is monotone increasing

and $\lim_{k\to\infty} g_k(x) = f(x) \geq \varphi(x)$.

So $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

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Hence $\lim m(A_n) = \infty$.

But, for each n,

$$\int f_n\,dx \geq \int g_n\,dx > a\, m(A_n).$$

So $\liminf \int f_n\,dx = \infty$

So $\int \varphi\,dx \leq \liminf \int f_n\,dx$.

Case 2: $\int \varphi\,dx < \infty$.

Write $B = \{x : \varphi(x) > 0\}$.

Then $m(B) < \infty$.

Let M be the largest value of φ,

and if $0 < \epsilon < 1$,

write $B_n = \{x : g_k(x) > (1-\epsilon)\varphi(x),\ k \geq n\}$,

where g_k is as defined above.

Then the sets B_n are measurable,

$B_n \subseteq B_{n+1}$ for each n,

and $\bigcup_{n=1}^{\infty} B_n \supseteq B$.

So $\{B - B_n\}$ is a decreasing sequence of sets,

$$\bigcap_{n\geq 1}^{\infty} (B - B_n) = \phi.$$

As $m(B) < \infty$, there exists N such that $m(B - B_n) < \epsilon$ for all $n \geq N$.

So if $n \geq N$,

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$$\int g_n\,dx \geq \int_{B_n} g_n\,dx \geq (1-\epsilon)\int_{B_n} \varphi\,dx$$

$$= (1-\epsilon)\left(\int_B \varphi\,dx - \int_{B-B_n} \varphi\,dx\right) \quad [\because \text{Th. } 3.2.1]$$

$$\geq (1-\epsilon)\int \varphi\,dx - \int_{B-B_n} \varphi\,dx$$

$$\geq \int \varphi\,dx - \epsilon\int \varphi\,dx - \epsilon M.$$

since ϵ is arbitrary,

$$\liminf \int g_n\,dx \geq \int \varphi\,dx,$$

and since $f_n \geq g_n$,

(2) proved.

Theorem 3.2.4: Lebesgue's Monotone Convergence Theorem

Let $\{f_n, n=1,2,\dots\}$ be a sequence of non-negative measurable functions such that $\{f_n(x)\}$ is monotone increasing for each x.

Let $f = \lim f_n$. Then $\int f\,dx = \lim \int f_n\,dx$.

Proof:

Fatou's Lemma gives

$$\int f\,dx = \int \liminf f_n\,dx \leq \liminf \int f_n\,dx. \longrightarrow (1)$$

but $f \geq f_n$ by hypothesis,

so by th: 3.2.2(i),

$$\int f\,dx \geq \int f_n\,dx,$$

Hence $\int f\,dx \geq \limsup f_n\,dx \longrightarrow (2)$

from (1) & (2), it is proved.

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Theorem 3.2.5:

Let f be a non-negative measurable function. Then there exists a sequence $\{\varphi_n\}$ of measurable simple functions such that, for each x, $\varphi_n(x) \uparrow f(x)$.

Proof:

Write, for each n,

$$E_{nk} = \{x : (k-1)/2^n < f(x) \le k/2^n\},$$

$k = 1, 2, \ldots, n2^n$, and $F_n = \{x : f(x) > n\}$.

put ~~F_n~~,

$$\varphi_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{nk}} + n \chi_{F_n}.$$

Then the functions φ_n are measurable simple functions.

Also, the range of f giving φ_{n+1} is a refinement of that giving φ_n,

$$\varphi_{n+1}(x) \ge \varphi_n(x) \text{ for each } x.$$

If $f(x)$ is finite,

$$x \in CF_n \text{ for all large } n,$$

and then $|f(x) - \varphi_n(x)| \le 2^{-n}$.

So $\varphi_n(x) \uparrow f(x)$.

If $f(x) = \infty$,

then $x \in \bigcap_{n=1}^{\infty} F_n$,

so $\varphi_n(x) = n$ for all n,

and again $\varphi_n(x) \uparrow f(x)$.

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Corollary:

$$\lim \int \varphi_n \, dx = \int f \, dx,$$

Where φ_n and f are as in Theorem 3.1.5.

Theorem 3.2.6:

Let f and g be non-negative measurable functions.

Then $\int f dx + \int g \, dx = \int (f+g) \, dx. \longrightarrow (1)$

Proof: Consider (1) for measurable simple fns. φ & ψ.

Let the values of φ be $a_1, a_2, \dots, a_n$ taken on sets $A_1, \dots, A_n$,

let the values of ψ be $b_1, \dots, b_m$ taken on sets $B_1, \dots, B_m$.

Then the simple function $\varphi + \psi$ has the value $a_i + b_j$ on the measurable set $A_i \cap B_j$,

so from Theorem 3.2.1 (i),

we obtain,

$$\int_{A_i \cap B_j} (\varphi + \psi) \, dx = \int_{A_i \cap B_j} \varphi \, dx + \int_{A_i \cap B_j} \psi \, dx. \longrightarrow (2)$$

But the union of the nm disjoint sets $A_i \cap B_j$ is $\mathbb{R}$, so theorem 3.2.1 (ii), applied to both sides of (2),

$$\int (\varphi + \psi) \, dx = \int \varphi \, dx + \int \psi \, dx. \longrightarrow (3).$$

Let f and g be any non-negative measurable functions.

Let $\{\varphi_n\}, \{\psi_n\}$ be sequences of measurable simple functions,

$$\varphi_n \uparrow f, \ \psi_n \uparrow g.$$

Then $\varphi_n + \psi_n \uparrow f+g$.

but by ③,

$$\int (\varphi_n + \psi_n)\,dx = \int \varphi_n\,dx + \int \psi_n\,dx.$$

Taking $n \to \infty$,

from theorem 3.2.4,

$$\int (f+g)\,dx = \int f\,dx + \int g\,dx.$$

Theorem 3.2.7:

Let $\{f_n\}$ be a sequence of non-negative measurable functions. Then

$$\int \sum_{n=1}^{\infty} f_n\,dx = \sum_{n=1}^{\infty} \int f_n\,dx. \longrightarrow ①$$

Proof:

By induction, previous theorem ① applies to a sum of n functions.

So if $S_n = \sum_{i=1}^{n} f_i$,

then $\int S_n\,dx = \sum_{i=1}^{n} \int f_i\,dx$.

But $S_n \uparrow f = \sum_{i=1}^{\infty} f_i$

∴ ① proved.

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The General integral

Definition : Positive and Negative Parts of f

If $f(x)$ is any real function,

$f^{+}(x) = \max(f(x), 0)$, — positive part of f

$f^{-1}(x) = \max(-f(x), 0)$ — Negative part of f.

Theorem 3.2.8 :

(i) $f = f^{+} - f^{-1}$; $|f| = f^{+} + f^{-1}$, $f^{+}, f^{-1} \geq 0$,

(ii) f is measurable if and only if f^{+} and f^{-1} are both measurable.

Integrable function

If f is a measurable function and $\int f^{+} dx < \infty$, $\int f^{-1} dx < \infty$, we say that f is integrable and its integral is given by

$$\int f\,dx = \int f^{+} dx - \int f^{-} dx.$$

Note: a measurable function f is integrable if and only if, $|f|$ is integrable,

Then $\int |f|\,dx = \int f^{+} dx + \int f^{-} dx$.

Integrable over E :

If E is a measurable set, f is a measurable function, and $\chi_E f$ is integrable, we say that f is integrable over E, its is given by $\int_E f\,dx = \int f \chi_E\,dx$. [(i.e) $f \in L(E)$]

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Definition: If f is a measurable function such that atleast one of $\int f^+ dx$, $\int f^- dx$ is finite,

Then $\int f dx = \int f^+ dx - \int f^- dx$.

Theorem 3.2.9:

Let f and g be integrable functions.

(i) af is integrable, and $\int af\,dx = a\int f\,dx$.

(ii) $f+g$ is integrable, and $\int (f+g)\,dx = \int f\,dx + \int g\,dx$.

(iii) If $f=0$ a.e., then $\int f\,dx = 0$.

(iv) If $f \leq g$ a.e., then $\int f\,dx \leq \int g\,dx$.

(V) If A and B are disjoint measurable sets,

then $\int_A f\,dx + \int_B f\,dx = \int_{A\cup B} f\,dx$.

Proof:

(i) If $a \geq 0$.

Then $(af)^+ = af^+$,

$(af)^- = af^-$.

So $\int (af)^+ dx < \infty$ &

$\int (af)^- dx < \infty$.

So af is integrable

and $\int af\,dx = \int af^+ dx - \int af^- dx = a\int f\,dx$.

If $a=-1$,

then $(-f)^+ = f^-$

$(-f)^- = f^+$

So $-f$ is integrable and

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$\int(-f)dx = \int f^{-}dx - \int f^{+}dx = -\int f\,dx.$

but for $a<0$, $af = -|a|f$,

so $\int af\,dx = -\int |a|f\,dx = -|a|\int f\,dx = a\int f\,dx.$

hence proved.

(ii) $(f+g)^{+} \le f^{+} + g^{+}$,

$(f+g)^{-} \le f^{-} + g^{-}$,

So $f+g$ is integrable.

$(f+g)^{+} - (f+g)^{-} = f+g = f^{+} + g^{+} - f^{-} - g^{-}$

So $(f+g)^{+} + f^{-} + g^{-} = (f+g)^{-} + f^{+} + g^{+}.$

Appling Theorem 3.2.6 both sides,

we get this result.

(iii) $f^{+} = 0$ a.e

and $f^{-} = 0$ a.e

$\int f^{+}dx = \int f^{-}dx = 0$ [$\because f = 0$ a.e iff $\int f\,dx = 0$]

(iv) $g = f + (g-f)$

$\int g\,dx = \int f\,dx + \int (g-f)^{+}dx - \int (g-f)^{-}dx.$

but $(g-f)^{-} = 0$ a.e [$\because$ (iii)]

it proves.

(v) $\chi_{A\cup B} = \chi_A + \chi_B$

It is proved. [$\because f = 0$ a.e iff $\int f\,dx = 0$ (ii)]

Example:

Show that f and g are measurable, $|f| \leq |g|$ a.e., and g is integrable, then f is integrable.

Soln: If $|f| \leq |g|$.

Then $f^+ \leq |g|$

So $\int f^+ \leq \int |g| \, dx < \infty$

IIIly for f^-.

Example: Show that if f is an integrable function, then $|\int f dx| \leq \int |f| dx$. When does equality occur?

Soln: $|f| - f \geq 0$.

So $\int |f| \, dx \geq \int f \, dx$.

also $|f| + f \geq 0$

So $\int |f| dx \geq -\int f \, dx$.

Hence $\int |f| \, dx \geq |\int f \, dx|$.

Necessary Condition for equality:

If $\int f dx \geq 0$,

Then $\int |f| \, dx = \int f dx$

(i.e.) $\int (|f| - f) \, dx = 0$.

$|f| = f$ a.e. [by previous examples]

If $\int f dx < 0$ then $\int |f| dx = \int (-f) \, dx$.

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(i.e) $\int (|f| + f)\,dx = 0$

so $|f| = -f$ a.e.

Hence $f \geq 0$ a.e (or)

$f \leq 0$ a.e is a necessary condition.

Theorem 3.2.10: Lebesgue's Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$, where g is integrable, and let $\lim f_n = f$ a.e. Then f is integrable

and $\lim \int f_n\,dx = \int f\,dx$. $\longrightarrow$ (1)

Proof:

For each n,

$|f_n| \leq g$,

we have $|f| \leq g$ a.e., [$\because$ if f & g are measurable, g is integrable then f is integrable]

so f_n and f are integrable

Also, $\{g + f_n\}$ is a sequence of non-negative measurable functions,

so by Fatou's Lemma

$\liminf \int (g + f_n)\,dx \geq \int \liminf (g + f_n)\,dx$.

So $\int g\,dx + \liminf \int f_n\,dx \geq \int g\,dx + \int f\,dx$.

but $\int g\,dx$ is finite

so $\liminf \int f_n\,dx \geq \int f\,dx$. $\longrightarrow$ (2)

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Again, $\{g-f_n\}$ is also a sequence of non-negative measurable functions,

So $\liminf \int (g-f_n)\,dx \geq \int \lim\inf (g-f_n)\,dx$.

So $\int g\,dx - \limsup \int f_n\,dx \geq \int g\,dx - \int f\,dx$.

So $\limsup \int f_n\,dx \leq \int f\,dx \leq \liminf \int f_n\,dx$

from ① & ②, it is proved.

<u>Theorem 3.2.11:</u>

Let $\{f_n\}$ be a sequence of integrable functions

Such that $\sum_{n=1}^{\infty} \int |f_n|\,dx < \infty \longrightarrow ①$

Then the series $\sum_{n=1}^{\infty} f_n(x)$ Converges a.e., its Sum $f(x)$ is integrable and

$$\int f\,dx = \sum_{n=1}^{\infty} \int f_n\,dx \longrightarrow ②.$$

<u>Proof:</u> Let $\varphi(x) = \sum_{n=1}^{\infty} |f_n|$.

Then by theorem 3.2.7,

$$\int \varphi\,dx < \infty,$$

So φ is finite-valued a.e. [$\because$ f is integrable Then f is finite valued a.e]

It follows that $\sum_{n=1}^{\infty} f_n(x)$ is absolutely Convergent a.e., its Sum $f(x)$ is defined a.e., and $|f| \leq \varphi$.

So f is integrable.

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Take $g_n(x) = \sum_{i=1}^{n} f_i(x)$.

Then $|g_n(x)| \le \varphi(x)$

& $\lim g_n(x) = f(x)$ a.e.

So by theorem 3.2.10,

$$\lim \int g_n \, dx = \int f \, dx$$

$$\therefore \int f dx = \sum_{n=1}^{\infty} \int f_n \, dx.$$

Theorem 3.2.12:

If f is continuous on the finite interval $[a, b]$, then f is integrable, and

$F(x) = \int_a^x f(t) \, dt$, $(a < x < b)$ is a differentiable function such that $F'(x) = x$.

Proof: Given f is continuous.

So it is measurable and $|f|$ is bounded.

So if f is integrable on $[a, b]$.

If $a < x < b$ we have $x+h \in (a, b)$, $\forall$ small h,

and $F(x+h) - F(x) = \int_x^{x+h} f(t) \, dt$.

$$\int_x^{x+h} f(t) \, dt = h f(\xi), \quad \xi = x + \theta h, \; 0 \le \theta \le 1.$$

[$\because$ If f is measurable, $m(E) < \infty$ & $A \le f \le B$ on E
Then $A m(E) \le \int_E f dx \le B m(E)$]

So if $h \ne 0$, dividing h and putting $h \to 0$ we get the result.

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Example: Show that if $\alpha > 1$,

$$\int_0^1 \frac{x \sin x}{1+(nx)^\alpha}\, dx = O(n^{-1}) \text{ as } n \to \infty.$$

Soln: We have to show that

$$\lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1+(nx)^\alpha}\, dx = 0.$$

clearly $\lim_{n\to\infty} \frac{nx \sin x}{1+(nx)^\alpha} = 0$,

Theorem 3.2.10 applies to the sequence

$$f_n(x) = \frac{nx \sin x}{1+(nx)^\alpha}, \quad n = 1, 2, \dots$$

Consider $h(x) = 1 + (nx)^\alpha - nx^{3/2}$.

So $h(0) = 1$, $h(1) = 1 + n^\alpha - n$.

For $1 < \alpha \leq 3/2$, h has no ~~sta~~ stationary point in $[0,1]$, for all large n;

for $\alpha > 3/2$ it has a stationary point at which its value is easily seen to approach 1 for large n.

So for large n, $h(x) > 0$ on $[0,1]$

and so $\left| \frac{nx \sin x}{1+(nx)^\alpha} \right| \leq \frac{1}{\sqrt{x}}$.

$$\therefore \lim_{n\to\infty} \int_0^1 \frac{nx \sin x}{1+(nx)^\alpha}\, dx = 0.$$

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Integration of Series

Example 1: Show that $\int_0^1 \frac{x^{1/3}}{1-x} \log \frac{1}{x}\, dx = 9 \sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$

Soln:

$$\frac{x^{1/3}}{1-x} \log \frac{1}{x} = x^{1/3} \log 1/x \sum_{n=0}^{\infty} x^n, \quad (0<x<1)$$

[∵ Th 3.2.7]

gives $$\int_0^1 \frac{x^{1/3}}{1-x} \log \frac{1}{x}\, dx = \sum_{n=0}^{\infty} \int_0^1 x^{n+1/3} \log \frac{1}{x}\, dx$$

$$= \sum_{n=0}^{\infty} \frac{9}{(3n+4)^2}$$

Example 2:

Show that $\int_0^{\infty} \frac{\sin t}{e^t - x}\, dt = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2+1}, \quad -1 \le x \le 1.$

Soln: The integrand $= \lim_{N\to\infty} \sum_{n=0}^{N} x^n \sin t\, e^{-(n+1)t}$, but $t \geq 0$,

$$\left| \sum_{n=0}^{N} x^n \sin t\, e^{-(n+1)t} \right| \le t e^{-t} \frac{1 - x^{N+1} e^{-(N+1)t}}{1 - x e^{-t}}$$

$$\le \frac{2t}{e^t - x}$$

an integrable fn.

So theorem 3.2.10 applies to the sequence of partial sum

giving $$\int_0^{\infty} \frac{\sin t}{e^t - x}\, dt = \sum_{n=0}^{\infty} x^n \int_0^{\infty} e^{-(n+1)t} \sin t\, dt$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{1+(n+1)^2}.$$

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Example 3:

Show that $\int_0^1 \sin x \log x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)(2n)!}$

Soln:

$$\sin x \log x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \log x = \sum_{n=0}^{\infty} f_n(x) \text{ say.}$$

but $\int_0^1 |f_n(x)| \, dx = (-1)^{n+1} \int_0^1 f_n(x) \, dx$

$$= \frac{1}{(2n+2)(2n+2)!}$$

So by theorem 3.2.11,

$$\int_0^1 \sin x \log x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)(2n)!}.$$

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Riemann and Lebesgue integrals

We consider the Riemann-integral of a bounded function f over a finite interval $[a, b]$.

Let $a = \xi_0 < \xi_1 < \dots < \xi_n = b$ be a partition D of $[a, b]$.

Write $S_D = \sum_{i=1}^{n} M_i (\xi_i - \xi_{i-1})$,

where $M_i = \sup f$ in $[\xi_{i-1}, \xi_i]$, $i = 1, \dots, n$.

$|||^{ly}$ replacing M_i by m_i equal to $\inf f$ over the corresponding interval,

we obtain $s_D = \sum_{i=1}^{n} m_i (\xi_i - \xi_{i-1})$.

Then f is said to be Riemann integrable over $[a, b]$

if given $\epsilon > 0$,

there exists D such that $S_D - s_D < \epsilon$.

here we have $\inf S_D = \sup s_D$,

where the infimum and supremum are taken over all partitions D of $[a, b]$,

We write the common value as $R\int_a^b f\,dx$.

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Theorem 3.2.13:

If f is Riemann integrable and bounded over the finite interval $[a,b]$, Then f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

Pf: Let $\{D_n\}$ be a sequence of partitions such that, for each n, $S_{D_n} - s_{D_n} < 1/n$

It is easily seen that

$$S_{D_n} = \int_a^b u_n\,dx \quad \& \quad s_{D_n} = \int_a^b l_n\,dx,$$

where u_n & l_n are step functions,

$$u_n \geq f \geq l_n.$$

Define $u_n = M_i$ on (ξ_{i-1}, ξ_i)

Let u_n be the average of the values M_i corresponding to the intervals ending at that point.

Write $U = \inf_n u_n$ & $L = \sup_n l_n$.

$$[x : U(x) - L(x) > 0] = \bigcup_{k=1}^{\infty} [x : U(x) - L(x) > 1/k]$$

Now if $U - L > 1/k$,

then $u_n - l_n > 1/k$ for each n.

So if $m[x : U(x) - L(x) > 1/k] = a$

then $\int (u_n - l_n)\,dx > a/k$

and so $a/k < 1/n$ for each n.

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So $a = 0$.

Hence $U - L \leq 1/k$ a.e for each k,

So $U = L$ a.e.

But u_n, l_n and hence U, L are measurable.

Also $L \leq f \leq U$,

So f is measurable and being bounded, is integrable.

clearly $\int_a^b l_n \, dx \leq \int_a^b f \, dx \leq \int_a^b u_n \, dx$,

and taking $n \to \infty$,

we get $R\int_a^b f dx = \int_a^b f \, dx$.

Theorem 3.2.14:

Let f be a bounded function defined on the finite interval $[a,b]$, then f is Riemann integrable over $[a,b]$ if and only if it is Continuous a.e.

proof:

If f is Riemann integrable over $[a, b]$.

Suppose that $U(x) = f(x) = L(x)$,

Where x is not a partition point of any D_n,

the D_n being Chosen as before.

Then f is Continuous at x;

for otherwise there would exist $\epsilon > 0$

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and a sequence $\{x_k\}$,

$\lim x_k = x$, such that for each k,

$|f(x_k) - f(x)| \geq \epsilon$.

But then $U(x) \geq L(x) + \epsilon$.

Now, the set of all partition points of the D_n is countable and so has measure zero, and the set $[x : U(x) \neq L(x)]$ has measure zero. [by Theorem 3.2.13]

So f is continuous a.e.

<u>Converse part:</u>

If f is continuous a.e.

Choose a sequence $\{D_n\}$ of partitions of $[a,b]$ such that, for each n, D_{n+1} contains the partition points of D_n and such that the length of the largest interval of D_n tends to zero as $n \to \infty$.

Then if u_n, l_n are the corresponding step fns. as in Theorem 3.2.13,

we have $u_{n+1} \leq u_n$ & $l_{n+1} \geq l_n$ for each n.

Let $U = \lim u_n$ & $L = \lim l_n$.

If f is continuous at x.

Then, given $\epsilon > 0$,

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there exists $\delta > 0$ such that

$$\sup f - \inf f < \epsilon,$$

where the supremum & infimum are taken over $(x-\delta, x+\delta)$.

For all n sufficiently large, an interval of D_n containing x will lie in $(x-\delta, x+\delta)$,

So $u_n(x) - l_n(x) < \epsilon$.

But ϵ is arbitrary, so $U(x) = L(x)$.

So $U = L$ a.e.

Then, by theorem 3.2.10,

$$\lim \int u_n \, dx = \int U \, dx = \int L \, dx = \lim \int l_n \, dx$$

$\therefore f$ is Riemann integrable.

Riemann integrable:

If for each a & b, f is bounded and Riemann integrable on $[a, b]$ and $\lim\limits_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f \, dx$ exists, then f is said to be Riemann integrable on $(-\infty, \infty)$.

It is denoted by $R \int_{-\infty}^{\infty} f \, dx$.

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Theorem 3.2.15:

Let f be bounded and let f and $|f|$ be Riemann integrable on $(-\infty, \infty)$. Then f is integrable and

$$\int_{-\infty}^{\infty} f\,dx = R\int_{-\infty}^{\infty} f\,dx.$$

Proof:

From Theorem 3.2.13,

$$\int_a^b |f|\,dx = R\int_a^b |f|\,dx$$

$$\leq R\int_{-\infty}^{\infty} |f|\,dx. \quad \text{for all } a \,\&\, b.$$

So f is integrable.

From Theorem 3.2.13 again gives.

$$\int_a^b f\,dx = R\int_a^b f\,dx.$$

Theorem 3.2.16:

Let f be bounded and measurable on a finite interval $[a,b]$ and let $\epsilon > 0$. Then there exist

(i) a step function h such that $\int_a^b |f-h|\,dx < \epsilon$,

(ii) a continuous function g such that g vanishes outside a finite interval and

$$\int_a^b |f-g|\,dx < \epsilon \longrightarrow (1)$$

Proof:

(i) As $f = f^{+} - f^{-1}$,

We may assume throughout that $f \geq 0$.

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Now $\int_a^b f\,dx = \sup \int_a^b \varphi\,dx$,

where $\varphi \leq f$, φ simple and measurable.

So we may assume that f is a simple measurable function, with $f = 0$ outside $[a, b]$.

$$\text{So } f = \sum_{i=1}^{n} a_i \chi_{E_i} \longrightarrow (2)$$

$$\text{With } \bigcup_{i=1}^{n} E_i = [a, b]$$

Let $\epsilon' = \epsilon / nM$ where $M = \sup f$ on $[a, b]$,

M may be positive.

For each of the measurable sets E_i there exist open intervals $I_1, I_2, \ldots, I_k$ such that,

$$\text{if } G = \bigcup_{r=1}^{k} I_r,$$

then $m(E_i \Delta G) < \epsilon'$.

But χ_G is a step function such that

$$\int |\chi_{E_i} - \chi_G|\,dx = m(E_i \Delta G) < \epsilon'.$$

Construct such step functions h_i, say, for each E_i appearing in (2),

$$\text{Then } \int_a^b |f - \sum_{i=1}^{n} a_i h_i|\,dx < \sum_{i=1}^{n} a_i \epsilon'$$

$$\leq nM\epsilon' = \epsilon.$$

But $\sum_{i=1}^{n} a_i h_i$ is a step function.

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(ii) From (i) there exists a step function h vanishing outside a finite interval, such that

$$\int_a^b |f-h|\,dx < \epsilon/2.$$

The proof is completed by constructing a continuous function g such that

$$\int |h-g|\,dx < \epsilon/2$$ and

such that $g(x)=0$ whenever $h(x)=0$.

Let $h = \sum_{i=1}^{n} a_i \chi_{E_i}$,

where E_i is the finite interval (c_i, d_i), $i=1,\ldots,n$.

As in (i), it is sufficient to show that each χ_{E_i} may be approximated like (1).

If $\epsilon < 2(d_i - c_i)$ and

define $g=1$ on $(c_i+\epsilon/4,\ d_i-\epsilon/4)$,

$g=0$ on $\mathcal{C}(c_i, d_i)$

Extend g by linearity to $(c_i, c_i+\epsilon/4)$ and $(d_i-\epsilon/4, d_i)$ as in fig. to get a continuous function.

so $\int |\chi_{E_i} - g|\,dx < \epsilon/2$

hence (ii) proved.

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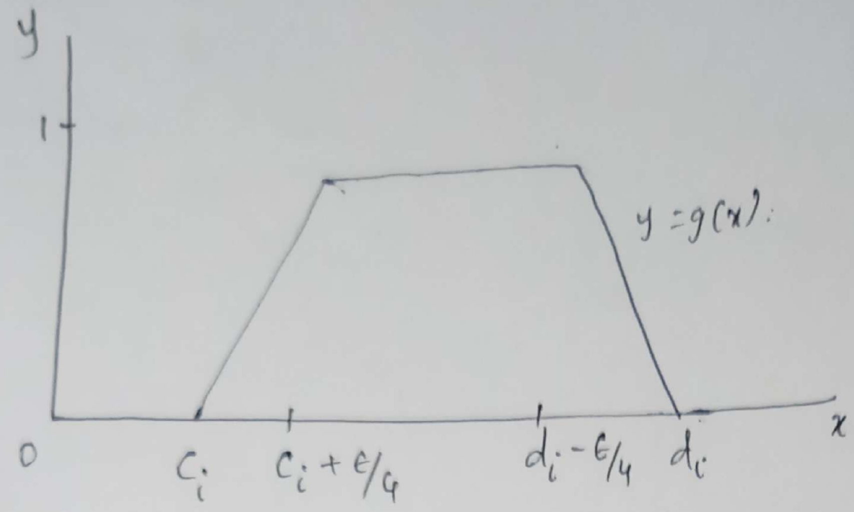

Corollary:

This theorem hold if f is integrable over $[a,b]$

We may assume $f \geq 0$.

UNIT IV

Measures and Outer Measures

RING:

A class of sets $\mathcal{R}$, of some fixed space is called a ring if whenever $E \in \mathfrak{R}$ and $F \in \mathfrak{R}$ then $E \cup F$ and E-F belong to R.

Example 1: The class of finite unions of intervals of the form [a, b) forms a ring.

σ-ring:

A ring is called a σ-ring if it is closed under the formation of countable unions.

Example 2:

Show that every algebra is a ring and every σ-algebra a σ-ring but that the converse is not true.

Theorem 4.1:

There exist a smallest ring and a smallest σ-ring containing a given class of subsets of a space; it is referred as the generated ring and the generated σ-ring respectively.

Proof:

In the proof of the Theorem 3.1.7, if we replace 'algebra' into 'σ-algebra' we get the proof of this theorem.

Notation:

S($\mathcal{R}$) - σ-ring S generated by the ring R

H($\mathcal{R}$) – for the class consisting of S($\mathcal{R}$) together with all subsets of the sets of S($\mathcal{R}$) .

Hereditary.

A class of sets with this property, namely that every subset of one of its members belongs to the class, is said to **hereditary.**

H($\mathcal{R}$) is a σ-ring and is the smallest hereditary σ-ring containing R.

H($\mathcal{R}$)= H(S($\mathcal{R}$))= H(H($\mathcal{R}$))

Definition : A set function μ defined on a ring R is a **measure** if

(i) μ is non-negative,

(ii) $\mu(\phi)=0$,

(iii) For any sequence $\{A_n\}$ of disjoint sets of R such that $\bigcup_{n=1}^{\infty} A_n \in \Re$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

If R is a σ-ring, the condition $\bigcup_{n=1}^{\infty} A_n \in \Re$ is clearly redundant.

Complete:

A measure μ on R is complete if whenever $E \in R$, $F \subseteq E$ and $F \subseteq E$ and $\mu(E)=0$, then $F \in R$.

σ-finite:

A measure μ on R is σ-finite if, for every set $E \in R$, we have $E = \bigcup_{n=1}^{\infty} E_n$ for some sequence $\{E_n\}$ such that $E_n \in R$ and $\mu(E_n) < \infty$ for each n.

Example : show that Lebesgue measure m defined on M, the class of measurable sets of R, is σ-finite and complete.

Outer Measure:

If R is a ring, a set function μ^* defined on the class $H(\mathcal{R})$ is an outer measure if

(i) μ^* is non-negative,

(ii) if $A \subseteq B$, then $\mu^*(A) \le \mu^*(B)$

(iii) $\mu^*(\phi) = 0$,

(iv) For any sequence $\{A_n\}$ of sets of $H(\mathcal{R})$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n),$$

(i.e.,) μ^* is countably subadditive.

Example: If $A, B \in R$ and $A \subseteq B$ then $\mu(A) \le \mu(B)$.

EXTENSION OF A MEASURE

Theorem 4.2:

Let {A_i} be a sequence in a ring T, then there is a sequence {B_i} of disjoint sets of R such that $B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N, so that

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

Proof:

Define {B_i} inductively by $B_i = A_i$, $B_n = A_n - \bigcup_{i=1}^{n-1} B_i$ for n>1.

Clearly $B_i \in R$ and $B_i \subseteq A_i$ for each i.

Also, as B_n and $\bigcup_{i=1}^{n-1} B_i$ are disjoint

We have $B_n \cap B_m = \phi$ for n>m.

Finally we have $B_i = A_i$ and if $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$

That implies $\cup\left(\bigcup_{i=1}^{k} B_i\right) = \left(A_{k+1} - \bigcup_{i=1}^{k} B_i\right) \cup \bigcup_{i=1}^{k} B_i$

$$= A_{k+1} \cup \bigcup_{i=1}^{k} B_i = A_{k+1} \cup \bigcup_{i=1}^{k} A_i$$

As required.

Example: Show that $H(R) = \left[E : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in R\right]$

Solution: It is easily checked that the right-hand side defines a class of sets which is hereditary, contains R, and is a σ-ring. So it contains H($\mathcal{R}$).

But if $E_n \in R$ for each n,

We have

$$\bigcup_{n=1}^{\infty} E_n \in S(\Re)$$

And so each subset belong to H(R).

So it is proved.

Theorem 4.3:

If μ is a measure on a ring R and if the set function μ* is defined on H(R) by

$$\mu^*(E) = \inf\left[\sum_{n=1}^{\infty}\mu(E_n) : E_n \in R, n = 1, 2, \ldots, E \subseteq \bigcup_{n=1}^{\infty} E_n\right] \quad \text{-------- (1)}$$

$$\mu^* = inf\left[\sum_{n=1}^{\infty}\mu(E_n): E_n \epsilon R,\ \ n = 1,2, \ldots ,\qquad E \subseteq \bigcup_{n=1}^{\infty} E_n\right]$$

Then (i) $E \in \Re,\ \mu^*(E)=\mu(E)$,

(ii) μ* is an outer measure on H(R).

Proof:

(i) If $E \in \Re$, (i) gives $\mu^*(E) \le \mu(E)$.

Suppose that $E \in \Re$,

And $E \subseteq \bigcup_{n=1}^{\infty} E_n$ where $E_n \in \Re$,

By theorem 4.2 we may replace the sequence $\{E_i \cap E\}$ by a sequence $\{F_i\}$ of disjoint sets of R,

Such that $F_i \subseteq E_i \cap E$ and $\bigcup_{i=1}^{\infty} F_i = E$.

Then by previous example, $\mu(F_i) \le \mu(E_i)$ for each i.

So $\mu(E) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty}\mu(F_i) \le \mu(E_i)$

Therefore $\mu(E) \le \mu^*(E)$.

So $\mu^*(E)=\mu(E)$.

(ii) $\mu^*(\phi)=\mu(\phi)$ by (i)

the only other property of an outer measure which is not immediate, namely countable subadditivity, is shown as m*.

$\{E_i\}$ is a sequence sets in H(R).

From the definition of μ*,for each $\varepsilon > 0$,

We can find for each i a sequence $\{E_{i,j}\}$ of sets of R such that

$$E_i \subseteq \bigcup_{j=1}^{\infty} E_{i,j} \text{ and } \sum_{j=1}^{\infty} \mu(E_{i,j}) \le \mu^*(E_i) + \varepsilon / 2^i .$$

The sets $E_{i,j}$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$,

So $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(E_{i,j}) \le \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$

But ε is arbitrary.

μ* -**Measurable:**

Let μ* be an outer measure on H(R). Then $E \in H(R)$ is μ*-measurable if for each $A \in H(R)$

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap CE)$$

Theorem 4.4 :

Let μ* be an outer measure on H(R) and let S* denote the class of μ*-measurable sets. Then S* is a σ-ring and μ* restricted to S* is a complete measure.

Proof:

S* is closed under countable unions.

It remains to be shown that if

$$E, F \in S^* \text{ then } E\text{-}F \in S^*.$$

Let $A \in H(R)$

and we can write A as the union of the four disjoint sets

$$A_1 = A - (E \bigcup F)$$

$$A_2 = A \cap E \cap F$$

$$A_3 = A \cap (F - E)$$

$$A_3 = A \cap (E - F)$$

Since F is measurable,

$$\mu^*(A) = \mu^*(A_1 \cup A_4) + \mu^*(A_2 \cup A_3) \quad \text{-------------- (1)}$$

[since $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap CE)$]

Replacing A in $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap CE)$ **by** $A_1 \cup A_4$ and using the fact that E is measurable gives

$$\mu^*(A_1 \cup A_4) = \mu^*(A_1) + \mu^*(A_4) \quad \text{--------------------- (2)}$$

Replacing A in $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap CE)$ **by** $A_1 \cup A_2 \cup A_3$ and using the fact that F is measurable gives

$$\mu^*(A_1 \cup A_2 \cup A_3) = \mu^*(A_1) + \mu^*(A_2 \cup A_3) \quad \text{--------(3)}$$

From (1), (2) & (3) we have

$$\mu^*(A) = \mu^*(A_4) + \mu^*(A_1 \cup A_2 \cup A_3),$$

Which is the condition for E-F to be measurable.

Suppose that $\{E_i\}$ is a sequence of disjoint sets in S*.

Then we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i).$$

So μ* is a measure on the σ-ring S*.

Also every set $E \in H(R)$ such that μ*(E)=0 is μ*-measurable,

For if $A \in H(R)$,

$$\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap CE)$$
$$\le \mu^*(E) + \mu^*(A) = \mu^*(A)$$

So E is μ*-measurable,

In particular if $E \in S^*$ and μ*(E)=0 and $F \subseteq E$ then it follows that $F \in S^*$.

So μ* is a complete measure on S*.

Theorem 4.5: Let μ* be an outer measure on H(R) defined by μ on R, then S* contains S(R), the σ-ring generated by R.

Proof:

Since S* is σ-ring it is sufficient to show that

$$R \subseteq S^*.$$

If $E \in R$, $A \in H(R)$,

and ε>0, then by the definition of μ* in theorem 3 (1) there exists a sequence $\{E_n\}$ of sets of R such that

$$A \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and}$$

$$\mu^*(A) + \varepsilon \ge \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n \cap E) + \sum_{n=1}^{\infty} \mu(E_n \cap CE)$$

As μ is a measure.

So $\mu^*(A) + \varepsilon \ge \mu^*(A \cap E) + \mu^*(A \cap CE)$

But ε is arbitrary so

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap CE)$$

The opposite inequality is obvious, so $E \in S^*$.

It gives the result.

Example:

Show that if μ is a σ-finite measure on R, then the extension $\bar{\mu}$ of μ to S* is also σ-finite.

Solution: Let $E \in S^*$.

Then by the definition of $\bar{\mu}$ there is a sequence $\{E_n\}$ of set R such that

$$\bar{\mu}(E) \leq \sum_{n=1}^{\infty} \mu(E_n) .$$

But each E_n is, by hypothesis, the union of a sequence $\{E_{n,i}, i=1,2,\ldots\}$ of set R such that

$\mu(E_{n,i}) < \infty$ for each n and i.

So $\bar{\mu}(E) \leq \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu(E_{n,i})$,

and so E is the union of a countable collection of sets of finite $\bar{\mu}$-measure.

UNIQUENESS OF THE EXTENSION:

Theorem 4.6:

The outer measure μ* on H(R) defined by μ on R as in Theorem 4.3, and the corresponding outer measure defined by $\bar{\mu}$ on S(R) and $\bar{\mu}$ on S* are the same.

Proof:

We first observe that the outer measure β* defined by a measure β on a σ-ring T satisfies, for $E \in H(T)$

$$\beta^*(E) = \inf\left[\beta(F); E \subseteq F \in \Im\right] \text{ ----------- (1)}$$

This is the case since

$$\beta^*(E) = \inf\left[\sum_{n=1}^{\infty}\beta(E_n); E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in \mathfrak{I}\right],$$

and replacing the sets E_n by disjoint sets $F_n \in \mathfrak{I}$, such that $F_n \subseteq E_n$ and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n,$$

We get $\sum_{n=1}^{\infty}\beta(E_n) \geq \sum_{n=1}^{\infty}\beta(F_n) = \beta\left(\bigcup_{n=1}^{\infty} F_n\right) \geq \beta^*(E)$

So (1) follows.

Since H(R)=H(S(R))=H(S*),

the outer measures to be considered have the same domain of definition.

As $\mu = \bar{\mu}$ on R,

$$\begin{aligned}
\mu^*(E) &= \inf\left[\sum_{n=1}^{\infty}\mu(E_n) : E \subseteq \bigcup_{n=1}^{\infty} E_n, E_n \in R\right] \\
&\geq \inf\left[\sum_{n=1}^{\infty}\bar{\mu}(F_n) : E \subseteq \bigcup_{n=1}^{\infty} F_n, F_n \in S(R)\right] \\
&= \inf\left[\sum_{n=1}^{\infty}\bar{\mu}(F) : E \subseteq F \in S(R)\right] \text{ by (1)} \\
&\geq \inf\left[\sum_{n=1}^{\infty}\bar{\mu}(F) : E \subseteq F \in S^*\right] \text{as } S^* \supseteq S(R) \\
&\geq \mu^*(E)
\end{aligned}$$

So equality holds throughout and so by (1) the outer measures are equal.

Corollary: since outer measure on H(R) determines the measurable sets and their measures, the measure and measurable sets obtained by extending, as in Theorem 4.3, μ on R, $\bar{\mu}$ on S(R) and $\bar{\mu}$ on S* are the same, namely $\bar{\mu}$ on S*.

Theorem 4.7: If μ is a σ-finite measure on a ring R, then it has a unique extension to the σ-ring S(R).

Proof: By theorem 4.3,

$\bar{\mu}$ on S(R) is an extension of μ.

Suppose that ν is a measure on S(R) such that μ= ν on R;

We wish to show that $\bar{\mu}=\nu$ on S(R).

If $E \in S(R)$ and ε>0, ∃ $[E_n]$, $E_n \in R$, $E \subseteq \bigcup_{n=1}^{\infty} E_n$

Such that $\bar{\mu}(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(E_n)$.

But $A = \bigcup_{n=1}^{\infty} E_n$

By Theorem 4.2, may be written as the union of disjoint sets F_n, $F_n \subseteq E_n$, $F_n \in R$;

We get

$$\bar{\mu}(E) + \varepsilon \geq \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} \nu(F_n) = \nu(A) \geq \nu(E)$$

So $\bar{\mu}(E) \geq \nu(E)$

Suppose that $E \in S(R)$, $\bar{\mu}(E) < \infty$ and ε>0 then as above there exists $A \supseteq E$ such that $\bar{\mu}(A) < \bar{\mu}(E) + \varepsilon$ where $A = \bigcup_{n=1}^{\infty} F_n$, the sets F_n being disjoint sets of R,

So that $\bar{\mu}(A) = \nu(A)$.

So $\bar{\mu}(E) \leq \bar{\mu}(A) = \nu(E) + \nu(A - E)$.

But, by the first part,

$$\nu(A - E) \leq \bar{\mu}(A - E),$$

Also since $\bar{\mu}(E) < \infty$.

We have $\bar{\mu}(A - E) < \varepsilon$.

So $\bar{\mu}(E) \leq \nu(E) + \varepsilon$

Hence $\bar{\mu}(E) = \nu(E)$ if $\bar{\mu}(E) < \infty$.

As μ is σ-finite, for each $E \in S(R)$

We have $E \subseteq \bigcup_{n=1}^{\infty} E_n$

Where, for each n, $E_n \in R$ and $\mu(E_n) < \infty$.

Then we may write $E = \bigcup_{n=1}^{\infty} F_n$

Where F_n are disjoint sets of R and $\mu(F_n) < \infty$.

So $\bar{\mu}(E) = \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} \nu(F_n) = \nu(E)$.

COMPLETION OF A MEASURE

Theorem 4.8:

If μ is a measure on a σ-ring S, then the class $\bar{S}$ of sets of the form $E \Delta N$ for any sets E, N such that $E \in S$ while N is contained in some set in S of zero measure, is a σ-ring , and the set function $\bar{\mu}$ defined by $\bar{\mu}(E\Delta N) = \mu(E)$ is a complete measure on $\bar{S}$.

Proof: It is convenient to have two different descriptions of the sets of $\bar{S}$ so we prove the set theoretic identity

$$E\Delta N = (E - M) \cup (M \cap (E\Delta N) \quad \text{-------} \ (1)$$

For any sets E, M, N such that $M \supseteq N$.

Let $x \in E\Delta N$, then if $x \in M$ we have

$$x \in M \cap (E\Delta N),$$

While if $x \in CM$ we have $x \in CN$ so $x \in E - N$

And hence $x \in E - M$.

To get the opposite inclusion in (1),

Suppose that x belongs to the right-hand side.

If $x \in M \cap (E\Delta N)$, then $x \in E\Delta N$;

If $x \in E - M$ we have $x \in E - N \subseteq E\Delta N$.

Let $D \in \bar{S}$, $D = E\Delta N$, as above with $N \subseteq M \in S$ with $\mu(M) = 0$.

Then by (1), $D = F \cup A$ where $F \cap A = \phi$ and $F \in S$ and $A \subseteq M \in S$ with $\mu(M) = 0$,

And since for F, A disjoint we have $F \cup A = F\Delta A$ the two characterizations of the sets of $\bar{S}$ are equivalent.

Now if $D_i \in \bar{S}, i = 1, 2, \ldots.$

On writing $D_i = F_i \cup A_i$

We see that $\bigcup_{i=1}^{\infty} D_i \in \bar{S}$.

If $D_1 = E_1 \Delta N_1$ and $D_2 = E_2 \Delta N_2$ belong to S

We have

$$D_1 \Delta D_2 = (E_1 \Delta E_2) \Delta (N_1 \Delta N_2).$$

So $D_1 \Delta D_2 \in \bar{S}$ and so $D_1 - D_2 = (D_1 \cup D_2) \Delta D_2 \in \bar{S}$.

So $\bar{S}$ is a σ-ring.

Also $D_1 \Delta D_2 = \phi$ only if $E_1 \Delta E_2 = N_1 \Delta N_2$.

So if $E_1 \Delta N_1 = E_2 \Delta N_2$,

We have $\mu(E_1 \Delta E_2) = 0$

And hence $\mu(E_1) = \mu(E_2)$.

So $\bar{\mu}$ is unambiguously defined.

Also $\bar{\mu}$ is a measure; for clearly $\bar{\mu}(\phi) = 0$,

And if {Di} is a sequence of disjoint sets of $\bar{S}$, $D_i = F_i \cup A_i$,say, in the notation used above,

So that $F_i \cup A_j = \phi$ for all i and j, then

$$\bar{\mu}\left(\bigcup D_i\right) = \bar{\mu}\left(\bigcup F_i \bigcup \bigcup A_i\right) = \bar{\mu}\left(\bigcup F_i \Delta \bigcup A_i\right) = \mu\left(\bigcup F_i\right) = \sum \mu(F_i)$$

$$= \sum \bar{\mu}(F_i \bigcup A_i) = \sum \bar{\mu}(D_i)$$

So $\bar{\mu}$ is countably additive.

Finally μ is complete,

for let $D \subset D_0 \in \bar{S}$ where $\bar{\mu}(D_0) = 0$.

so $D_0 = E_0 \Delta N_0$ where $N_0 \subseteq M_0, E_0, M_0 \in S,\ \mu(E_0) = \mu(M_0) = 0$,

and so $D_0 \subseteq M_0' = E_0 \cup M_0 \in S,\ \mu(M_0') = 0$

then $D = E \Delta N$ with E=ϕ, N=D$\subseteq E_0 \cup M_0$

and so $D \in \bar{S}$.

Example: show that the extension $\bar{\mu}$ of Theorem 4.8 is unique in the sense that if μ' is a complete measure on a σ-ring , $S' \supseteq \bar{S}$ and $\mu' = \mu$ on S then $\mu' = \bar{\mu}$ on $\bar{S}$.

Solution: since μ' is complete it is easily seen that $S' \supseteq \bar{S}$.

For $D \in \bar{S}$

We have as above $D = F \cup A$;

A disjoint sets with $F \in S,\ A \subseteq M \in S$ with $\mu(M)=0$.

So $\mu'(D) = \mu'(F) + \mu'(A) = \bar{\mu}(D)$.

We call $\bar{\mu}$ on $\bar{S}$ the completion of μ on S.

Theorem 4.9:

The completion of a σ-finite measure is σ-finite.

Proof:

Let $D \in \bar{S}$.

As is theorem 4.8,

$$D = F \cup A \text{ where } F \in S \text{ and } \bar{\mu}(A) = 0.$$

So $F = \bigcup_{i=1}^{\infty} F_i$ where $\mu(F_i) < \infty$,

And hence $D = A \cup \bigcup_{i=1}^{\infty} F_i$ is a countable union of sets of finite $\bar{\mu}$-measure.

MEASURE SPACES

Measurable Space: A pair [[X, S]] where S is a σ-algebra of subsets of a space X, is called a measurable space. The sets of S are called measurable sets.

Measure space: [[X, S, μ]] is called a measure space if [[X, S]] is a measurable space and μ is a measure on S.

Example: [[R, M, m]] and [[R, B, m]] are measure spaces, where B denotes the Borel sets, and where in the second example m is restricted to B.

In the latter case m is called Borel measure on the real line.

Example: Let [[X, S]] be a measurable space and let $Y \in S$. Then if $S' = [B \cap Y : B \in S]$ we have that [[Y, S']] is a measurable space.

Theorem 4.10:

Let $\{E_i\}$ be a sequence of measurable sets. We have

(i) If $E_1 \subseteq E_2 \subseteq \ldots$, then $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$.

(ii) $E_1 \supseteq E_2 \supseteq \ldots$, and $\mu(E_i) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim \mu(E_n)$

Proof: Refer Unit III -Theorem 3.9.

Measurable: Let f be an extended real-valued function defined on X. Then f is said to be measurable if $\forall \alpha$, $[x : f(x) < \alpha] \in S$.

Example: Let [[X, S]] be a measurable space and let $X = \bigcup_{n=1}^{\infty} X_n$ where, for each n, $X_n \in S$ and $X_n \cap X_m = \phi$ for $n \neq m$. Write $S_n = [B \cap X_n : B \in S]$.

Show that f is measurable with respect to [[X, S]] only if , for each n, its restriction f_n to X_n is measurable with respect to [[X_n, S_n]], and conversely if, for each n, the functions f_n are measurable with respect to [[X_n, S_n]] and f is defined by $f(x)=f_n(x)$ when $x \in X_n$, then f is measurable with respect to [[X, S]].

Solution: For each α, $[x : f_n(x) > \alpha] = [x : f(x) > \alpha] \cap X_n$

So f_n is measurable with respect to the measurable space [[X_n, S_n]].

Conversely,

$$[x : f(x) > \alpha] = \bigcup_{n=1}^{\infty} [x : f_n(x) > \alpha].$$

Theorem 4.11:

The measurability of f is equivalent to

(i) f is measurable function

(ii) $\forall\alpha, [f(x) \geq \alpha] \in S$,

(iii) $\forall\alpha, [x : f(x) < \alpha] \in S$

(iv) $\forall\alpha, [x : f(x) \leq \alpha] \in S$

Proof:

Let f be measurable.

Then $[x : f(x) \geq \alpha] = \bigcap_{n=1}^{\infty} [x : f(x) > \alpha - \frac{1}{n}]$ is measurable;

So (i) => (ii).

Let $\forall\alpha, [f(x) \geq \alpha]$ be measurable.

Then $[x : f(x) < \alpha] = C[x : f(x) \geq \alpha]$

is measurable and (ii)=>(iii).

If (iii) holds, then

$[x : f(x) \leq \alpha] = \bigcap_{n=1}^{\infty} [x : f(x) < \alpha + \frac{1}{n}]$ is measurable;

So (iii)=> (iv).

Finally, if $[x : f(x) \leq \alpha]$ is measurable,

then its complement $[x : f(x) > \alpha]$ is measurable;

so (iv)=>(i) .

hence the theorem is proved.

Theorem 4.12:

If c is a real number and f,g measurable functions, then f+c, cf, f+g, g-f and fg are also measurable.

Proof:

For each α, $[x : f(x)+c>\alpha]=[x : f(x)>\alpha-c]$, a measurable set,

So f+c is measurable.

If c=0, cf is measurable. [since constant functions are measurable]

Otherwise, if c>0, $[x : cf(x)>\alpha]=[x : f(x)>c^{-1}\alpha]$, a measurable set.

Similarly if c<0 can also proved.

So cf is always measurable.

To show that f+g is measurable,

$x\in A=[x : f(x)+g(x)>\alpha]$ only if $f(x)>\alpha-g(x)$

that is, only if there exists a rational r_i such that $f(x)>r_i>\alpha-g(x)$, where $\{r_i, i=1,2,...\}$ is an enumeration of Q.

But then $g(x)>\alpha-r_i$

and so $x\in[x : f(x)>r_i]\cap[x : g(x)>\alpha-r_i]$

Hence $A\subseteq B=\bigcup_{i=1}^{\infty}\left([x : f(x)>r_i]\cap[x : g(x)>\alpha-r_i]\right)$, a measurable set.

Since A clearly contains B we have A=B and so f+g is measurable.

Then f-g=f+(-g) is also measurable.

Finally $fg=\frac{1}{2}\left((f+g)^2-(f-g)^2\right)$.

So it is sufficient to show that f^2 is measurable whenever f is.

If $\alpha<0$, $[x : f^2(x)>\alpha]=R$ is measurable.

If $\alpha\geq 0$, $[x : f^2(x)>\alpha]=[x : [x : f(x)<-\sqrt{\alpha}]f(x)>\sqrt{\alpha}]\cap R$, a measurable set.

Theorem 4.13:

If f_i is measurable, i=1,2,… then $\sup_{1\leq i\leq n} f_i$, $\inf_{1\leq i\leq n} f_i$, $\sup f_n$, $\inf f_n$, $\limsup f_n$ and $\liminf f_n$

are also measurable.

Proof:

(i) Since $[x : \sup_{1\le i\le n} f_i(x) > \alpha] = \bigcup_{i=1}^{n} [x : f_i(x) > \alpha]$,

we have $\sup_{1\le i\le n} f_i$ is measurable.

(ii) $\inf_{1\le i\le n} f_i = -\sup_{1\le i\le n}(-f_i)$, so it is measurable.

(iii) $[x : \sup\ f_n(x) > \alpha] = \bigcup_{n=1}^{\infty} [x : f_n(x) > \alpha]$, so sup f_n is measurable.

(iv) $\inf\ f_n = -\sup\ (-f_n)$, so inf f_n is measurable.

(v) $\lim\sup\ f_n = \inf(\sup_{i\ge n} f_i)$, a measurable function by (iii) & (iv).

(vi) $\lim\inf f_n = -\lim\sup(-f_n)$, so lim inf f_n is measurable.

Example: The limit of a pointwise convergent sequence of measurable functions is measurable.

Example: Let f=g a.e.(μ), where μ is a complete measure. If f is measurable, then g is also measurable.

INTEGRATION WITH RESPECT TO A MEASURE

Measurable simple function ϕ : A Measurable simple function ϕ is one taking a finite number of non-negative values, each on a measurable set; so if $a_1, \ldots, a_n$ are the distinct values of ϕ,

We have $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ where $A_i = [x : \phi(x) = a_i]$.

Then the integral of ϕ with respect to μ is given by

$$\int \phi d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

Definition:

Let f be measurable, $f : X \to [0, \infty]$. Then the integral of f is

$$\int f d\mu = \sup[\phi d\mu : \phi \le f, \phi \text{ is a measurable simple function}] .$$

Definition:

Let $E \in S$, and let f be a measurable function $f: E \to [0, \infty]$; then the integral of f over E is $\int_E f d\mu = \int f \chi_E d\mu$.

Theorem 4.14: Fatou's Lemma

Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$, then $\lim \inf \int f_n d\mu = \int \lim \inf f_n d\mu$.

Proof: Refer Theorem 3.2.3 .

Theorem 4.15: Lebesgue's Monotone Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$, such that $f_n(x) \uparrow$ for each x, and let $f = \lim f_n$. Then $\int f dx = \lim \int f_n d\mu$.

Proof: Refer Theorem 3.2.4.

Theorem 4.16:

Let f be a measurable function $f : X \to [0, \infty]$. Then there exists a sequence $\{\phi_n\}$ of measurable simple functions such that, for each x, $\phi_n(x) \uparrow f(x)$.

Proof: Refer Theorem 3.2.5.

Theorem 4.17:

Let $\{f_n\}$ be a sequence of measurable functions, $f_n : X \to [0, \infty]$; then

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu .$$

Proof: Refer Theorem 3.2.6.

Theorem 4.18:

Let [[X, S. μ]] be a measure space and f a non-negative measurable function. Then $\phi(E) = \int_E f d\mu$ is a measure on the measurable space [[X, S]]. If, in addition , $\int f d\mu < \infty$ then $\forall\ \varepsilon>0, \exists\ \delta>0$ such that, if $A \in S$ and $\mu(A)<\delta$, then $\phi(A)<\varepsilon$.

Proof:

The function φ is countably additive since, if $\{E_n\}$ is a sequence of disjoint sets of S,

$$\phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \int \chi \cup E_n f d\mu = \sum_{n=1}^{\infty} \int \chi E_n f d\mu$$

By theorem 4.17,

The other properties being obvious, φ is a measure on [[X, S]].

Write $f_n = \min(f, n)$.

Then f_n is measurable, $f_n \uparrow f$ and $\lim \int f_n d\mu = \int f d\mu$ by theorem 4.15.

So if $\int f d\mu < \infty$ then $\forall$ ε>0, $\exists$ N such that

$$\int f d\mu < \int f_N d\mu + \varepsilon/2 \ .$$

If $A \in S$ and $\mu(A) < \varepsilon/2N$ we have $\int f_N d\mu < \varepsilon/2$.

So take $\delta = \varepsilon / 2N$ to get

$$\int_A f d\mu = \int_A (f - f_N) d\mu + \int_A f_N d\mu$$

$$\leq \int (f - f_N) d\mu + \varepsilon / 2 < \varepsilon \ .$$

Definition : Integrable:

If f is measurable and both $\int f^+ d\mu$ and $\int f^- d\mu$ are finite, then f is said to be integrable, and the integral of f is $\int f^+ d\mu - \int f^- d\mu$.

So f is integrable if and only if |f| is integrable.

The notation $f \in L(X, \mu)$ is used to indicate that f belongs to the class of functions integrable with respect to μ.

The notation $\int_E f d\mu$ means $\int f \ \chi_E d\mu$, where $f \in L(X, \mu)$ and E in S.

If $f\chi_E$ is integrable we write $f \in L(X, \mu)$ or $f \in L(E)$.

Definition: we define $\int f d\mu = \int f^{+} d\mu - \int f^{-} d\mu$ provided atleast one of the integrals on the right-hand side is finite.

Theorem 4.19:

Let f and g be integrable functions and let a and b be constants. Then af+bg is integrable and $\int (af + bg) d\mu = a\int f d\mu + b\int g d\mu$. If f=g a.e., then $\int f d\mu = \int g d\mu$.

Proof: Refer Theorem 3.2.9.

Theorem 4.20:

Let f be integrable, then $\left|\int f d\mu\right| \le \int |f| d\mu$ with equality, if and only if, $f \ge 0$ a.e. or $f \le 0$ a.e.

Proof: Refer unit 3 second part.

Theorem 21 : Legesgue's Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \le g$ where g is an integrable function, and lim f_n=f a.e. Then f is integrable, $\lim \int f_n d\mu = \int f d\mu$, and $\lim \int |f_n - f| d\mu = 0$.

Proof: Refer Theorem 3.2.10 and the example.

Theorem 4.22:

Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$. Then $\sum_{n=1}^{\infty} f_n$ converges a.e., its sum f, is integrable and $\int f d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

Proof : Refer theorem 3.2.11.

UNIT V

SIGNED MEASURES AND THEIR DERIVATIVES

SIGNED MEASURES AND THE HAHN DECOMPOSITION

Signed Measure:

A set function ν defined on a measurable space [[X, S]] is said to be a signed measure if the values of ν are extended real numbers and

(i) ν takes at most one of the values ∞, - ∞,

(ii) ν(ϕ)=0,

(iii) $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$ if $E_i \cap E_j = \phi$ for $i \neq j$, where if the left-hand side is infinite, the series on the right-hand side has sum ∞ or - ∞ as the case may be.

Clearly, every measure is a signed measure.

Example 1: Show that if $\phi(E) = \int_E f d\mu$ where $\int f d\mu$ is defined, then φ is a signed measure.

Solution :

We have either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$

So (i) of Definition 1 follows.

(ii) is trivial.

Let $\{E_i\}$ be a sequence of disjoint sets of S and for $E \in S$

write $\phi^+(E) = \int_E f^+ d\mu$, $\phi^-(E) = \int_E f^- d\mu$

so by Theorem 4.18, ϕ^+ and ϕ^- are measures.

Then $\phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \phi^+\left(\bigcup_{i=1}^{\infty} E_i\right) - \phi^-\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \phi^+(E_i) - \sum_{i=1}^{\infty} \phi^-(E_i) = \sum_{i=1}^{\infty} \phi(E_i)$

As we cannot get $\infty - \infty$.

Positive set:

A is a positive set with respect to the signed measure ν on [[X, S]] if $A \in S$ and $\nu(E) \geq 0$ for each measurable subset E of A. We will omit 'with respect to ν' if the signed measure is obvious from the context.

Clearly φ is a positive set with respect to every signed measure. Also $\nu(A) \geq 0$ is necessary but not in general sufficient for A to be a positive set with respect to ν.

Example 2: If A is a positive set with respect to ν and if , for $E \in S$, μ(E)=ν(E∩A) , then μ is a measure.

Negative set:

A is a negative set with respect to ν if it is a positive set with respect to – ν.

Null set:

A is a null set with respect to ν, or a ν-null set, if it is both a positive and a negative set with respect to ν.

Equivalently, A is a ν-null set if $A \in S$ and ν(E)=0 for all $E \in S$, $E \subseteq A$.

Example 3: If A is a positive set with respect to ν, then every measurable subset of A is a positive set. The same holds for negative sets and null sets.

Theorem 1:

A countable union of sets positive with respect to a signed measure ν is a positive set.

Proof:

Let $\{A_n\}$ be a sequence of positive sets.

Then theorem 4.2,

We have $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$

Where the sets $B_n \in S$, $B_n \subseteq A_n$ and $B_n \cap B_m = \varnothing$ if $n \neq m$.

Now let $E \subseteq \bigcup_{n=1}^{\infty} A_n$

Then $E = \bigcup_{n=1}^{\infty} (E \cap B_n)$,

So $\nu(E) = \bigcup_{n=1}^{\infty} \nu(E \cap B_n) \geq 0$,

As $E \cap B_n$ is a positive set for each n by Example 3.

So $\bigcup_{n=1}^{\infty} A_n$ is a positive set.

Corollary : A countable union of negative or of null sets is , respectivey, a negative or a null set.

Theorem 5.2:

Let ν be a signed measure on [[X, S]]. Let $E \in S$ and ν(E)>0. Then there exists A, a set positive with respect to ν, such that $A \subseteq E$ and ν(A)>0.

Proof:

If E contains no set of negative ν-measure, then E is a positive set and A=E gives the result.

Otherwise,

There exists $n \in N$ such that there exists $B \in S$, $B \subseteq E$ and ν(B)<-1/n.

Let n_1 be the smallest such integer and E_1 a corresponding measurable subset of E with $\nu(E_1)<-1/n_1$.

Let n_k be the smallest positive integer such that there is a measurable subset E_k of

$$E - \bigcup_{i=1}^{k-1} E_i \text{ with } \nu(E_k) < -1/n_k .$$

From the construction, $n_1 \leq n_2 \leq ...$

and we have a corresponding sequence $\{E_i\}$ of disjoint subsets of E.

If the process stops, at n_m say,

and $C = E - \bigcup_{i=1}^{m} E_i$,

then C is a positive set,and ν(C)>0,

for ν(C)=0 would imply that $\nu(E) = \sum_{i=1}^{m} \nu(E_i) < 0$.

So C is the desired set.

If the process does not stop,

Put $A = E - \bigcup_{k=1}^{\infty} E_k$;

We wish to show that A is a positive set.

We have $\nu(E) = \nu(A) + \nu\left(\bigcup_{k=1}^{\infty} E_k\right)$. --------- (1)

But ν cannot take both the values ∞, -∞, ν(E)>0

and $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) < 0$.

So the second term on the right-hand side of (1) is finite.

So $\sum_{k=1}^{\infty} \nu(E_k) > -\infty$;

Hence $\sum_{k=1}^{\infty} 1/n_k < \infty$ and

In particular $\lim_{k\to\infty} n_k = \infty$, and $n_k > 1$ for $k>k_{0,}$ say.

So Let $B \in S$, $B \subseteq A$ and $k>k_0$.

Then

$$B \subseteq E - \bigcup_{i=1}^{k} E_i$$

So $\nu(B) \geq -\frac{1}{n_k - 1}$ by the definition of n_k.-------- (2)

But (2) holds for all $k>k_0$, so letting $k \to \infty$ we have $\nu(B) \geq 0$

And so A is a positive set.

As before, ν(A)=0 would imply ν(E)<0, so ν(A)>0.

Theorem 5.3:

Let ν be a signed measure on [[X, S]]. Then there exists a positive set A and a negative set B such that $A \cup B = X, A \cap B = \emptyset$. The pair A, B is said to be a Hahn decomposition of X with respect to ν. It is unique to the extent that if A_1, B_1 and A_2, B_2 are Hahn decompositions of X with respect to ν, then $A_1 \Delta A_2$ is a ν-null set.

Proof:

We may suppose that $\nu < \infty$ on S, for otherwise we consider $-\nu$,

the result of the theorem for $-\nu$ implying the result for ν.

Let λ=sup[ν (C): C a positive set], so $\lambda \geq 0$.

Let $\{A_i\}$ be a sequence of positive sets such that $\lambda = \lim \nu(A_i)$.

By Theorem 5.1,

$A = \bigcup_{i=1}^{\infty} A_i$ is a positive set, and from the definition of λ, $\lambda \geq \nu(A)$.

But $A - A_i \subseteq A$ and hence is a positive set.

So, for each i,

$$\nu(A) = \nu(A_i) + \nu(A - A_i) \geq \nu(A_i)$$

So $\nu(A) \geq \lim \nu(A_i) = \lambda$.

Hence $\nu(A) = \lambda$, that is, the value λ is achieved on a positive set.

Write B=CA.

Then if B contains a set D of positive ν-measure,

We have $0 < \nu(D) < \infty$.

So by Theorem 5.2, D contains a positive set E such that $0 < \nu(E) < \infty$.

But then $\nu(A \cup E) = \nu(A) + \nu(E) > \lambda$, contradicting the definition of λ.

So $\nu(D) \leq 0$ and B is a negative set and A, B form a Hahn decomposition.

For the last part note that $A_1 - A_2 = A_1 \cap B_2$ and hence is a positive and negative set and a null set.

Similarly $A_1 - A_2$ is a null set, and so $A_1 \Delta A_2$ is null.

The Jordan Decomposition

Mutually singular:

Let ν_1, ν_2 be measures on [[X, S]]. Then ν_1 and ν_2 are said to be mutually singular if, for some $A \in S$, $\nu_2(A) = \nu_1(CA) = 0$, and we write this as $\nu_1 \perp \nu_2$.

Example 4:

Let μ be a measure and let the measures ν_1, ν_2 be given by

$$\nu_1(E) = \mu(A \cap E), \nu_2(E) = \mu(B \cap E),$$

where $\mu(A \cap E) = 0$ and $E, A, B \in S.$ show that $\nu_1 \perp \nu_2$.

Solution :

$$\nu_1(B) = \mu(A \cap B) = 0, \nu_2(CB) = \mu(\emptyset) = 0.$$

Theorem 5.4:

Let ν be a signed measure on [[X,S]]. Then there exist measure ν^+ and ν^- on [[X,S]] such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$. The measures ν^+ and ν^- are uniquely defined by ν, and $\nu = \nu^+ - \nu^-$ is said to be the Jordan decomposition of ν.

Proof:

Let A, B be a Hahn decomposition of X with respect to ν, and define ν^+ and ν^- by

$$\nu^+(E) = \mu(E \cap A), \nu^-(E) = \mu(E \cap B) \text{ -------- } (1)$$

for $E \in S.$

then ν^+ and ν^- are measures by Example 2,

and $\nu^+(B) = \nu^-(A) = 0$.

So $\nu^+ \perp \nu^-$.

Also, for $E \in S.$

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = \nu^+(E) - \nu^-(E)$$

So $\nu = \nu^+ - \nu^-$.

If we complete the proof when we show that the decomposition is unique.

Let $\nu = \nu_1 - \nu_2$ be any decomposition of ν into mutually singular measures.

Then we have $X = A \cup B,$ where $B = CA$ and $\nu_1(B) = \nu_2(A) = 0.$

Let $D \subseteq A,$ then $\nu(D) = \nu_1(D) - \nu_2(D) = \nu_1(D) \geq 0$, So A is positive set with respect to ν.

Similarly B is a negative set.

For each $E \in S$, we have $\nu_1(E) = \nu_1(E \cap A)$

and $\nu_2(E) = -\nu(E \cap B)$

so every such decomposition of ν is obtained from a Hahn decomposition of X,as in (1).

So it is enough to show that if A, B are two Hahn decompositions then the measures obtained as in (1) as the same.

$$\nu(A \cup A') = \nu(A \cap A') + \nu(A \,\Delta\, A') = \nu(A \cap A')$$

By theorem 5.3.

For each $E \in S$,

As $A \cup A'$ is a positive set we have

$$\nu(E \cap (A \cap A') \leq \nu(E \cap A) \leq \nu(E \cap (A \cup A')$$

And $\nu(E \cap (A \cap A') \leq \nu(E \cap A') \leq \nu(E \cap (A \cup A')$

But the first and last terms in each of these inequalities are the same.

So $\nu(E \cap A) = \nu(E \cap A^1)$ and ν^+ defined in (1) is unique.

But the $\nu^- = \nu^+ - \nu$ is also unique.

Here we have note that the Hahn decomposition is of the space and isnot unique whereas the Jordan decomposition is of the signed measure and is unique.

Example 5:

Let [[X, S, μ]] be a measure space and let $\int f d\mu$ exist. Define ν by $\nu(E) = \int_E f d\mu, \text{ for } E \in S$. Find a Hahn decomposition with respect to ν and the Jordan decomposition of ν.

Solution: From example 1,

ν is a signed measure.

Let $A = [x : f(x) \geq 0], B = [x : f(x) < 0]$.

Then A, B form a Hahn decomposition, while ν^+, ν^- given by

$\nu^+ = \int_E f^+ d\mu, \nu^- = \int_E f^- d\mu$ form the Jordan decomposition.

Total Variation of a signed measure:

Total Variation of a signed measure ν is $|\nu| = \nu^+ + \nu^-$,

Where $\nu = \nu^+ + \nu^-$ is the Jordan decomposition of ν.

Clearly | ν | is a measure on [[X, S]], and for each $E \in S$, $|\nu(E)| \leq |\nu|(E)$.

Definition : A signed measure ν on [[X, S]] is σ-finite if $X = \bigcup_{n=1}^{\infty} X_n$

Where $X_n \in S$ and for each n,

$|\nu(X_n)| < \infty$.

Example 6:

Show that the signed measure ν is finite or σ-finite respectively if and only if , |ν| is or if and only if both ν^+ and ν^- are.

Solution:

Suppose |ν|<∞.

Then as ν^+ and ν^- are not both infinite we have $\nu^+(E) < \infty$ and $\nu^-(E) < \infty$.

Hence |ν|<∞,

Obviously, ν is finite if |ν| is.

The corresponding results on σ-finiteness are an immediate consequence.

The Radon-Nikodym Theorem

Absolutely continuous:

If μ, ν are signed measures on the measurable space [[X, S]] and ν(E)=0 whenever μ(E)=0, then we say that ν is absolutely continuous with respect to μ and we write $\nu \ll \mu$.

If μ, ν are signed measures on the measurable space [[X, S]] and $\nu(E)=0$ whenever

$|\mu|(E)=0$, then ν is absolutely continuous with respect to μ and we write $\nu \ll \mu$.

Example:

Show that the following conditions on the signed measures μ, ν on [[X, S]] are equivalent:

(i) $\nu \ll \mu$.

(ii) $|\nu| \ll |\mu|$.

(iii) $\nu^{+} \ll \mu$

(iv) $\nu^{-} \ll \mu$

Solution:

From the definition of absolutely continuous,

We see that $\nu \ll \mu$, if and only if $\nu \ll |\mu|$.

So we assume that $\mu \geq 0$.

As $|\nu| = \nu^{+} + \nu^{-}$,

We see that $|\nu| \ll \mu$ implies $\nu^{+} \ll \mu$ and

so $\nu \ll \mu$.

For the opposite implications,

Suppose that $\nu = \nu^{+} - \nu^{-}$ with a Hahn decomposition A, B.

Then if $\nu \ll \mu$ and $\mu(E)=0$ we have $\mu(E \cap A) = 0$

so $\nu^{+}(E)=0$

similarly $\nu^{-}(E)=0$.

So $|\nu|(E)=0$.

Example 8 : If μ is measure, $\int f d\mu$ exists and $\nu(E) = \int_E f d\mu$, then $\nu \ll \mu$.

Theorem 5.5: Radon - Nikodym Theorem

If [[X, S, μ]] is a σ-finite measure space and ν is a σ-finite measure on S such that, then there exists a finite-valued non-negative measurable function f on X such that for each $E \in S$, $\nu(E) = \int_E f d\mu$. Also f is unique in the sense that if $\nu(E) = \int_E g d\mu$ for each $E \in S$, then f=g a.e.(μ).

Proof:

Suppose that the result has been proved for finite measures. Then in the general case we have

Corollary 1: Theorem 5.5 can be extended to the case where ν is a σ-finite signed measure.

Corollary 2 : Theorem 5.5 can be further extended to allow μ to be signed measure, where by $\int_E f d\mu$ we then mean $\int_E f^+ d\mu - \int_E f^- d\mu$,provided this difference is not indeterminate. Any two such functions f and g are equal a.e. (|μ|).

Theorem 5.6:

Let μ be a signed measure on [[X, S]] and let ν be a finite-valued signed measure on [[X, S]] such that $\nu \ll \mu$; then given ε>0 there exists δ>0 such that |ν|(E)<ε whenever |μ|(E)<δ.

Example 9 : If μ and ν are signed measures on [[X, S]] and if ∀ ε>0, ∃ δ>0 such that whenever |μ|(E)<δ we have |ν|(E)<ε, then $\nu \ll \mu$.

Definition 10:

Let μ and ν be σ-finite signed measures on [[X, S]] and suppose that $\nu \ll \mu$. Then the Radon-Nikodym derivative d ν/dμ, of ν with respect to μ, is any measurable function f such that $\nu(E) = \int_E f\, d\mu$ for each $E \in S$, where if μ is a signed measure $\int f d\mu = \int f d\mu^+ - \int f d\mu^-$.

Theorem 5.7:

Lebesgue Decomposition Theorem

Let [[X, S, μ]] be a σ-finite measure space and ν a σ-finite measure on S. Then $\nu = \nu_0 + \nu_1$ where ν_0, ν_1 are measures on S such that $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$. This is the Lebesgue decomposition of the measure ν with respect to μ and it is unique.

Bounded Linear Functionals on L^p.

Normed Vector space:

Let V be a real vector space. Then V is a normed vector space if there is a function ||x|| defined for each $x \epsilon V$ such that

(i) $\forall x,\ \|x\| \geq 0$,
(ii) $\|x\| = 0$ if and only if x=0
(iii) $\|ax\| = |a|.||x||$ for any real number a and each $x \epsilon V$,
(iv) $\|x + y\| = \|x\| + \|y\|, \forall x, y \epsilon V$.

Linear functional :

A function G on the normal linear space V to the real numbers is a linear functional if $\forall x, y \epsilon V$ and a, $b \epsilon R$, we have

G(ax+by)=aG(x)+bG(y).

Bounded:

A linear functional G on the normed linear space V is bounded if ∍ K≥0 such that

$|G(x)| \geq \text{K}\|x\|, \forall x \ \epsilon \ V$.--------(1)

Then the norm of G, denoted by $\|G\|$, is the infimum of the numbers K for which (1) holds.

$$|G(x)| \leq \|G\|.\|x\|$$

Then dividing by $\|G\|$ we see that $\|G\| = sup[|G(x)|: \|x\| \leq 1]$

When dim V=0, $\|G\| = sup[|G(x)|: \|x\| = 1]$

Theorem 5.7:

Riesz Representation Theorem for $L^p, p > 1$

Let G be a bounded linear function on $L^p(X, \mu)$. Then there exists a unique element g of $L^q(X, \mu)$ such that

G(f)=$\int fg \ \ d\mu$ for each $f \epsilon L^p$

Where p,q are conjugate indices. Also

$\|G\| = \|g\|_q$.

Theorem 5.8:

Riesz Representation Theorem for L^1

Let [[X, S, μ]] be a σ-finite measure space and let G be a bounded linear functional on $\boldsymbol{L^1(X,\ \mu)}$. Then there exists a unique $g \in L^{\infty}(X,\ \mu)$ such that

$$G(f) = \int fg\ d\mu \text{ for each } f \in L^1(\mu).$$

Also, $\|G\| = \|g\|_{\infty}$